EXAMPLE 2. A horizontal line $y = c_2$ ($c_2 \neq 0$) is mapped by $w = 1/z$ onto the circle

$$u^2 + \left(v + \frac{1}{2c_2}\right)^2 = \left(\frac{1}{2c_2}\right)^2, \tag{6}$$

which is centered on the v axis and tangent to the u axis. Two special cases are shown in Fig. 111, where corresponding orientations of the lines and circles are also indicated.

EXAMPLE 3. When $w = 1/z$, the half plane $x \geq c_1$ ($c_1 > 0$) is mapped onto the disk

$$\left(u - \frac{1}{2c_1}\right)^2 + v^2 \leq \left(\frac{1}{2c_1}\right)^2. \tag{7}$$

For, according to Example 1, any line $x = c$ ($c \geq c_1$) is transformed into the circle

$$\left(u - \frac{1}{2c}\right)^2 + v^2 = \left(\frac{1}{2c}\right)^2. \tag{8}$$

Furthermore, as c increases through all values greater than c_1, the lines $x = c$ move to the right and the image circles (8) shrink in size. (See Fig. 112.) Since the lines $x = c$ pass through all points in the half plane $x \geq c_1$ and the circles (8) pass through all points in the disk (7), the mapping is established.

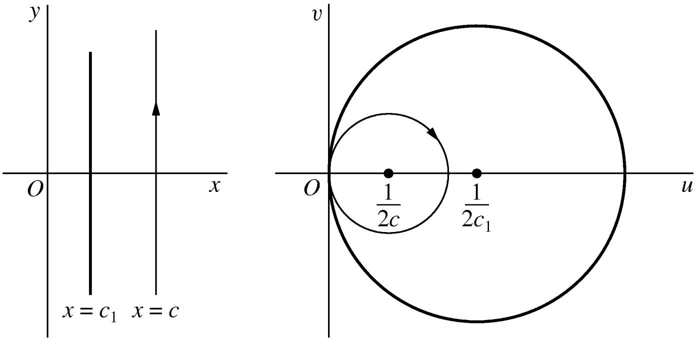

FIGURE 112
$w = 1/z$.

EXERCISES

1. In Sec. 92, point out how it follows from the first of equations (2) that when $w = 1/z$, the inequality $x \geq c_1$ ($c_1 > 0$) is satisfied if and only if inequality (7) holds. Thus give an alternative verification of the mapping established in Example 3, Sec. 92.

2. Show that when $c_1 < 0$, the image of the half plane $x < c_1$ under the transformation $w = 1/z$ is the interior of a circle. What is the image when $c_1 = 0$?

3. Show that the image of the half plane $y > c_2$ under the transformation $w = 1/z$ is the interior of a circle when $c_2 > 0$. Find the image when $c_2 < 0$ and when $c_2 = 0$.

4. Find the image of the infinite strip $0 < y < 1/(2c)$ under the transformation $w = 1/z$. Sketch the strip and its image.

Ans. $u^2 + (v + c)^2 > c^2,\ v < 0.$

5. Find the image of the region $x > 1$, $y > 0$ under the transformation $w = 1/z$.

Ans. $\left(u - \dfrac{1}{2}\right)^2 + v^2 < \left(\dfrac{1}{2}\right)^2,\ v < 0.$

6. Verify the mapping, where $w = 1/z$, of the regions and parts of the boundaries indicated in *(a)* Fig. 4, Appendix 2; *(b)* Fig. 5, Appendix 2.

7. Describe geometrically the transformation $w = 1/(z - 1)$.

8. Describe geometrically the transformation $w = i/z$. State why it transforms circles and lines into circles and lines.

9. Find the image of the semi-infinite strip $x > 0,\ 0 < y < 1$ when $w = i/z$. Sketch the strip and its image.

Ans. $\left(u - \dfrac{1}{2}\right)^2 + v^2 > \left(\dfrac{1}{2}\right)^2,\ u > 0,\ v > 0.$

10. By writing $w = \rho \exp(i\phi)$, show that the mapping $w = 1/z$ transforms the hyperbola $x^2 - y^2 = 1$ into the lemniscate $\rho^2 = \cos 2\phi$. (See Exercise 14, Sec. 5.)

11. Let the circle $|z| = 1$ have a positive, or counterclockwise, orientation. Determine the orientation of its image under the transformation $w = 1/z$.

12. Show that when a circle is transformed into a circle under the transformation $w = 1/z$, the center of the original circle is *never* mapped onto the center of the image circle.

13. Using the exponential form $z = re^{i\theta}$ of z, show that the transformation

$$w = z + \frac{1}{z},$$

which is the sum of the identity transformation and the transformation discussed in Secs. 91 and 92, maps circles $r = r_0$ onto ellipses with parametric representations

$$u = \left(r_0 + \frac{1}{r_0}\right)\cos\theta, \quad v = \left(r_0 - \frac{1}{r_0}\right)\sin\theta \qquad (0 \le \theta \le 2\pi)$$

and foci at the points $w = \pm 2$. Then show how it follows that this transformation maps the entire circle $|z| = 1$ onto the segment $-2 \le u \le 2$ of the u axis and the domain outside that circle onto the rest of the w plane.

14. *(a)* Write equation (3), Sec. 92, in the form

$$2Az\bar{z} + (B - Ci)z + (B + Ci)\bar{z} + 2D = 0,$$

where $z = x + iy$.

(b) Show that when $w = 1/z$, the result in part *(a)* becomes

$$2Dw\overline{w} + (B + Ci)w + (B - Ci)\overline{w} + 2A = 0.$$

Then show that if $w = u + iv$, this equation is the same as equation (4), Sec. 92.
Suggestion: In part *(a)*, use the relations (see Sec. 5)

$$x = \frac{z + \overline{z}}{2} \quad \text{and} \quad y = \frac{z - \overline{z}}{2i}.$$

93. LINEAR FRACTIONAL TRANSFORMATIONS

The transformation

$$w = \frac{az + b}{cz + d} \qquad (ad - bc \neq 0), \tag{1}$$

where a, b, c, and d are complex constants, is called a *linear fractional transformation*, or Möbius transformation. Observe that equation (1) can be written in the form

$$Azw + Bz + Cw + D = 0 \qquad (AD - BC \neq 0); \tag{2}$$

and, conversely, any equation of type (2) can be put in the form (1). Since this alternative form is linear in z and linear in w, another name for a linear fractional transformation is *bilinear transformation.*

When $c = 0$, the condition $ad - bc \neq 0$ with equation (1) becomes $ad \neq 0$; and we see that the transformation reduces to a nonconstant linear function. When $c \neq 0$, equation (1) can be written

$$w = \frac{a}{c} + \frac{bc - ad}{c} \cdot \frac{1}{cz + d} \qquad (ad - bc \neq 0). \tag{3}$$

So, once again, the condition $ad - bc \neq 0$ ensures that we do not have a constant function. The transformation $w = 1/z$ is evidently a special case of transformation (1) when $c \neq 0$.

Equation (3) reveals that when $c \neq 0$, a linear fractional transformation is a composition of the mappings.

$$Z = cz + d, \quad W = \frac{1}{Z}, \quad w = \frac{a}{c} + \frac{bc - ad}{c} W \qquad (ad - bc \neq 0).$$

It thus follows that, regardless of whether c is zero or nonzero, *any linear fractional transformation transforms circles and lines into circles and lines* because these special linear fractional transformations do. (See Secs. 90 and 92.)

Solving equation (1) for z, we find that

$$z = \frac{-dw + b}{cw - a} \qquad (ad - bc \neq 0). \tag{4}$$

EXERCISES

1. Find the linear fractional transformation that maps the points $z_1 = 2, z_2 = i, z_3 = -2$ onto the points $w_1 = 1, w_2 = i, w_3 = -1$.

Ans. $w = \dfrac{3z + 2i}{iz + 6}$.

2. Find the linear fractional transformation that maps the points $z_1 = -i, z_2 = 0, z_3 = i$ onto the points $w_1 = -1, w_2 = i, w_3 = 1$. Into what curve is the imaginary axis $x = 0$ transformed?

3. Find the bilinear transformation that maps the points $z_1 = \infty, z_2 = i, z_3 = 0$ onto the points $w_1 = 0, w_2 = i, w_3 = \infty$.

Ans. $w = -1/z$.

4. Find the bilinear transformation that maps distinct points z_1, z_2, z_3 onto the points $w_1 = 0, w_2 = 1, w_3 = \infty$.

Ans. $w = \dfrac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}$.

5. Show that a composition of two linear fractional transformations is again a linear fractional transformation, as stated in Sec. 93. To do this, consider two such transformations

$$T(z) = \frac{a_1 z + b_1}{c_1 z + d_1} \quad (a_1 d_1 - b_1 c_1 \neq 0)$$

and

$$S(z) = \frac{a_2 z + b_2}{c_2 z + d_2} \quad (a_2 d_2 - b_2 c_2 \neq 0).$$

Then show that the composition $S[T(z)]$ has the form

$$S[T(z)] = \frac{a_3 z + b_3}{c_3 z + d_3},$$

where

$$a_3 d_3 - b_3 c_3 = (a_1 d_1 - b_1 c_1)(a_2 d_2 - b_2 c_2) \neq 0.$$

6. A *fixed point* of a transformation $w = f(z)$ is a point z_0 such that $f(z_0) = z_0$. Show that every linear fractional transformation, with the exception of the identity transformation $w = z$, has at most two fixed points in the extended plane.

7. Find the fixed points (see Exercise 6) of the transformation

(a) $w = \dfrac{z - 1}{z + 1}$; *(b)* $w = \dfrac{6z - 9}{z}$.

Ans. (a) $z = \pm i$; *(b)* $z = 3$.

8. Modify equation (1), Sec. 94, for the case in which both z_2 and w_2 are the point at infinity. Then show that any linear fractional transformation must be of the form $w = az$ $(a \neq 0)$ when its fixed points (Exercise 6) are 0 and ∞.

9. Prove that if the origin is a fixed point (Exercise 6) of a linear fractional transformation, then the transformation can be written in the form

$$w = \frac{z}{cz + d} \qquad (d \neq 0).$$

10. Show that there is only one linear fractional transformation which maps three given distinct points z_1, z_2, and z_3 in the extended z plane onto three specified distinct points w_1, w_2, and w_3 in the extended w plane.

Suggestion: Let T and S be two such linear fractional transformations. Then, after pointing out why $S^{-1}[T(z_k)] = z_k$ $(k = 1, 2, 3)$, use the results in Exercises 5 and 6 to show that $S^{-1}[T(z)] = z$ for all z. Thus show that $T(z) = S(z)$ for all z.

11. With the aid of equation (1), Sec. 94, prove that if a linear fractional transformation maps the points of the x axis onto points of the u axis, then the coefficients in the transformation are all real, except possibly for a common complex factor. The converse statement is evident.

12. Let

$$T(z) = \frac{az + b}{cz + d} \qquad (ad - bc \neq 0)$$

be any linear fractional transformation other than $T(z) = z$. Show that

$$T^{-1} = T \quad \text{if and only if} \quad d = -a.$$

Suggestion: Write the equation $T^{-1}(z) = T(z)$ as

$$(a + d)[cz^2 + (d - a)z - b] = 0.$$

95. MAPPINGS OF THE UPPER HALF PLANE

Let us determine all linear fractional transformations that map the upper half plane $\operatorname{Im} z > 0$ onto the open disk $|w| < 1$ and the boundary $\operatorname{Im} z = 0$ of the half plane onto the boundary $|w| = 1$ of the disk (Fig. 113).

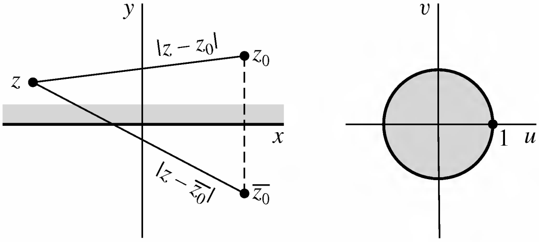

FIGURE 113

$$w = e^{i\alpha}\left(\frac{z - z_0}{z - \overline{z_0}}\right) \qquad (\operatorname{Im} z_0 > 0).$$

Keeping in mind that points on the line $\operatorname{Im} z = 0$ are to be transformed into points on the circle $|w| = 1$, we start by selecting the points $z = 0$, $z = 1$, and $z = \infty$ on the line and determining conditions on a linear fractional transformation

$$w = \frac{az + b}{cz + d} \qquad (ad - bc \neq 0) \tag{1}$$

which are necessary in order for the images of those points to have unit modulus.