

EXERCISES

1. In each case, show that any singular point of the function is a pole. Determine the order m of each pole, and find the corresponding residue B .

$$(a) \frac{z^2 + 2}{z - 1}; \quad (b) \left(\frac{z}{2z + 1} \right)^3; \quad (c) \frac{\exp z}{z^2 + \pi^2}.$$

Ans. (a) $m = 1, B = 3$; (b) $m = 3, B = -3/16$; (c) $m = 1, B = \pm i/2\pi$.

2. Show that

$$(a) \operatorname{Res}_{z=-1} \frac{z^{1/4}}{z + 1} = \frac{1 + i}{\sqrt{2}} \quad (|z| > 0, 0 < \arg z < 2\pi);$$

$$(b) \operatorname{Res}_{z=i} \frac{\operatorname{Log} z}{(z^2 + 1)^2} = \frac{\pi + 2i}{8};$$

$$(c) \operatorname{Res}_{z=i} \frac{z^{1/2}}{(z^2 + 1)^2} = \frac{1 - i}{8\sqrt{2}} \quad (|z| > 0, 0 < \arg z < 2\pi).$$

3. Find the value of the integral

$$\int_C \frac{3z^3 + 2}{(z - 1)(z^2 + 9)} dz,$$

taken counterclockwise around the circle (a) $|z - 2| = 2$; (b) $|z| = 4$.

Ans. (a) πi ; (b) $6\pi i$.

4. Find the value of the integral

$$\int_C \frac{dz}{z^3(z + 4)},$$

taken counterclockwise around the circle (a) $|z| = 2$; (b) $|z + 2| = 3$.

Ans. (a) $\pi i/32$; (b) 0.

5. Evaluate the integral

$$\int_C \frac{\cosh \pi z}{z(z^2 + 1)} dz$$

when C is the circle $|z| = 2$, described in the positive sense.

Ans. $4\pi i$.

6. Use the theorem in Sec. 71, involving a single residue, to evaluate the integral of $f(z)$ around the positively oriented circle $|z| = 3$ when

$$(a) f(z) = \frac{(3z + 2)^2}{z(z - 1)(2z + 5)}; \quad (b) f(z) = \frac{z^3(1 - 3z)}{(1 + z)(1 + 2z^4)}; \quad (c) f(z) = \frac{z^3 e^{1/z}}{1 + z^3}.$$

Ans. (a) $9\pi i$; (b) $-3\pi i$; (c) $2\pi i$.

7. Let z_0 be an isolated singular point of a function f and suppose that

$$f(z) = \frac{\phi(z)}{(z - z_0)^m},$$

where m is a positive integer and $\phi(z)$ is analytic and nonzero at z_0 . By applying the extended form (6), Sec. 51, of the Cauchy integral formula to the function $\phi(z)$,

show that

$$\operatorname{Res}_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!},$$

as stated in the theorem of Sec. 73.

Suggestion: Since there is a neighborhood $|z - z_0| < \varepsilon$ throughout which $\phi(z)$ is analytic (see Sec. 24), the contour used in the extended Cauchy integral formula can be the positively oriented circle $|z - z_0| = \varepsilon/2$.

75. ZEROS OF ANALYTIC FUNCTIONS

Zeros and poles of functions are closely related. In fact, we shall see in the next section how zeros can be a source of poles. We need, however, some preliminary results regarding zeros of analytic functions.

Suppose that a function f is analytic at a point z_0 . We know from Sec. 52 that all of the derivatives $f^{(n)}(z)$ ($n = 1, 2, \dots$) exist at z_0 . If $f(z_0) = 0$ and if there is a positive integer m such that $f^{(m)}(z_0) \neq 0$ and each derivative of lower order vanishes at z_0 , then f is said to have a *zero of order m* at z_0 . Our first theorem here provides a useful alternative characterization of zeros of order m .

Theorem 1. *Let a function f be analytic at a point z_0 . It has a zero of order m at z_0 if and only if there is a function g , which is analytic and nonzero at z_0 , such that*

$$(1) \quad f(z) = (z - z_0)^m g(z).$$

Both parts of the proof that follows use the fact (Sec. 57) that if a function is analytic at a point z_0 , then it must have a Taylor series representation in powers of $z - z_0$ which is valid throughout a neighborhood $|z - z_0| < \varepsilon$ of z_0 .

We start the first part of the proof by assuming that expression (1) holds and noting that since $g(z)$ is analytic at z_0 , it has a Taylor series representation

$$g(z) = g(z_0) + \frac{g'(z_0)}{1!}(z - z_0) + \frac{g''(z_0)}{2!}(z - z_0)^2 + \dots$$

in some neighborhood $|z - z_0| < \varepsilon$ of z_0 . Expression (1) thus takes the form

$$f(z) = g(z_0)(z - z_0)^m + \frac{g'(z_0)}{1!}(z - z_0)^{m+1} + \frac{g''(z_0)}{2!}(z - z_0)^{m+2} + \dots$$

when $|z - z_0| < \varepsilon$. Since this is actually a Taylor series expansion for $f(z)$, according to Theorem 1 in Sec. 66, it follows that

$$(2) \quad f(z_0) = f'(z_0) = f''(z_0) = \dots = f^{(m-1)}(z_0) = 0$$

It now follows from equation (2) that

$$\lim_{R \rightarrow \infty} \int_{-R}^R \frac{x^2}{x^6 + 1} dx = \frac{\pi}{3},$$

or

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{x^2}{x^6 + 1} dx = \frac{\pi}{3}.$$

Since the integrand here is even, we know from equation (7) in Sec. 78 that

$$(4) \quad \int_0^{\infty} \frac{x^2}{x^6 + 1} dx = \frac{\pi}{6}.$$

EXERCISES

Use residues to evaluate the improper integrals in Exercises 1 through 5.

$$1. \int_0^{\infty} \frac{dx}{x^2 + 1}.$$

Ans. $\pi/2$.

$$2. \int_0^{\infty} \frac{dx}{(x^2 + 1)^2}.$$

Ans. $\pi/4$.

$$3. \int_0^{\infty} \frac{dx}{x^4 + 1}.$$

Ans. $\pi/(2\sqrt{2})$.

$$4. \int_0^{\infty} \frac{x^2 dx}{(x^2 + 1)(x^2 + 4)}.$$

Ans. $\pi/6$.

$$5. \int_0^{\infty} \frac{x^2 dx}{(x^2 + 9)(x^2 + 4)^2}.$$

Ans. $\pi/200$.

Use residues to find the Cauchy principal values of the integrals in Exercises 6 and 7.

$$6. \int_{-\infty}^{\infty} \frac{dx}{x^2 + 2x + 2}.$$

$$7. \int_{-\infty}^{\infty} \frac{x dx}{(x^2 + 1)(x^2 + 2x + 2)}.$$

Ans. $-\pi/5$.

8. Use a residue and the contour shown in Fig. 95, where $R > 1$, to establish the integration formula

$$\int_0^{\infty} \frac{dx}{x^3 + 1} = \frac{2\pi}{3\sqrt{3}}.$$

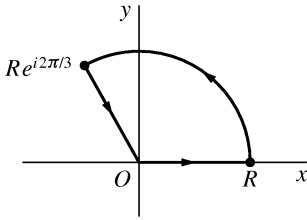


FIGURE 95

9. Let m and n be integers, where $0 \leq m < n$. Follow the steps below to derive the integration formula

$$\int_0^\infty \frac{x^{2m}}{x^{2n} + 1} dx = \frac{\pi}{2n} \csc\left(\frac{2m+1}{2n}\pi\right).$$

- (a) Show that the zeros of the polynomial $z^{2n} + 1$ lying above the real axis are

$$c_k = \exp\left[i\frac{(2k+1)\pi}{2n}\right] \quad (k = 0, 1, 2, \dots, n-1)$$

and that there are none on that axis.

- (b) With the aid of Theorem 2 in Sec. 76, show that

$$\operatorname{Res}_{z=c_k} \frac{z^{2m}}{z^{2n} + 1} = -\frac{1}{2n} e^{i(2k+1)\alpha} \quad (k = 0, 1, 2, \dots, n-1)$$

where c_k are the zeros found in part (a) and

$$\alpha = \frac{2m+1}{2n}\pi.$$

Then use the summation formula

$$\sum_{k=0}^{n-1} z^k = \frac{1 - z^n}{1 - z} \quad (z \neq 1)$$

(see Exercise 9, Sec. 8) to obtain the expression

$$2\pi i \sum_{k=0}^{n-1} \operatorname{Res}_{z=c_k} \frac{z^{2m}}{z^{2n} + 1} = \frac{\pi}{n \sin \alpha}.$$

- (c) Use the final result in part (b) to complete the derivation of the integration formula.

10. The integration formula

$$\int_0^\infty \frac{dx}{[(x^2 - a)^2 + 1]^2} = \frac{\pi}{8\sqrt{2}A^3} [(2a^2 + 3)\sqrt{A+a} + a\sqrt{A-a}],$$

where a is any real number and $A = \sqrt{a^2 + 1}$, arises in the theory of case-hardening of steel by means of radio-frequency heating.* Follow the steps below to derive it.

(a) Point out why the four zeros of the polynomial

$$q(z) = (z^2 - a)^2 + 1$$

are the square roots of the numbers $a \pm i$. Then, using the fact that the numbers

$$z_0 = \frac{1}{\sqrt{2}}(\sqrt{A+a} + i\sqrt{A-a})$$

and $-z_0$ are the square roots of $a + i$ (Exercise 5, Sec. 10), verify that $\pm \overline{z_0}$ are the square roots of $a - i$ and hence that z_0 and $-\overline{z_0}$ are the only zeros of $q(z)$ in the upper half plane $\text{Im } z \geq 0$.

(b) Using the method derived in Exercise 7, Sec. 76, and keeping in mind that $z_0^2 = a + i$ for purposes of simplification, show that the point z_0 in part (a) is a pole of order 2 of the function $f(z) = 1/[q(z)]^2$ and that the residue B_1 at z_0 can be written

$$B_1 = -\frac{q''(z_0)}{[q'(z_0)]^3} = \frac{a - i(2a^2 + 3)}{16A^2 z_0}.$$

After observing that $q'(-\overline{z}) = -\overline{q'(z)}$ and $q''(-\overline{z}) = \overline{q''(z)}$, use the same method to show that the point $-\overline{z_0}$ in part (a) is also a pole of order 2 of the function $f(z)$, with residue

$$B_2 = \left\{ \frac{\overline{q''(z_0)}}{[\overline{q'(z_0)}]^3} \right\} = -\overline{B_1}.$$

Then obtain the expression

$$B_1 + B_2 = \frac{1}{8A^2 i} \text{Im} \left[\frac{-a + i(2a^2 + 3)}{z_0} \right]$$

for the sum of these residues.

(c) Refer to part (a) and show that $|q(z)| \geq (R - |z_0|)^4$ if $|z| = R$, where $R > |z_0|$. Then, with the aid of the final result in part (b), complete the derivation of the integration formula.

80. IMPROPER INTEGRALS FROM FOURIER ANALYSIS

Residue theory can be useful in evaluating convergent improper integrals of the form

$$(1) \quad \int_{-\infty}^{\infty} f(x) \sin ax \, dx \quad \text{or} \quad \int_{-\infty}^{\infty} f(x) \cos ax \, dx,$$

*See pp. 359–364 of the book by Brown, Hoyler, and Bierwirth that is listed in Appendix 1.