If we replace the index of summation $n$ in the first of these series by $n-1$ and then interchange the two series, we arrive at an expansion having the same form as the one in the statement of Laurent's theorem (Sec. 60):

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} + \sum_{n=1}^{\infty} \frac{1}{z^n} \qquad (1 < |z| < 2). \tag{4}$$

Since there is only one Laurent series for $f(z)$ in the annulus $D_2$, expansion (4) is, in fact, *the* Laurent series for $f(z)$ there.

**EXAMPLE 5.** The representation of the function (1) in the unbounded domain $D_3$, where $2 < |z| < \infty$, is also a Laurent series. Since $|2/z| < 1$ when $z$ is in $D_3$, it is also true that $|1/z| < 1$. So if we write expression (1) as

$$f(z) = \frac{1}{2} \cdot \frac{1}{1-(1/z)} - \frac{1}{z} \cdot \frac{1}{1-(2/z)},$$

we find that

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} - \sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}} = \sum_{n=0}^{\infty} \frac{1-2^n}{z^{n+1}} \qquad (2 < |z| < \infty).$$

Replacing $n$ by $n-1$ in this last series then gives the standard form

$$f(z) = \sum_{n=1}^{\infty} \frac{1-2^{n-1}}{z^n} \qquad (2 < |z| < \infty) \tag{5}$$

used in Laurent's theorem in Sec. 60. Here, of course, all the $a_n$'s in that theorem are zero.

## EXERCISES

**1.** Find the Laurent series that represents the function

$$f(z) = z^2 \sin\left(\frac{1}{z^2}\right)$$

in the domain $0 < |z| < \infty$.

*Ans.* $1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot \frac{1}{z^{4n}}$.

**2.** Derive the Laurent series representation

$$\frac{e^z}{(z+1)^2} = \frac{1}{e}\left[\sum_{n=0}^{\infty} \frac{(z+1)^n}{(n+2)!} + \frac{1}{z+1} + \frac{1}{(z+1)^2}\right] \qquad (0 < |z+1| < \infty).$$

**3.** Find a representation for the function

$$f(z) = \frac{1}{1+z} = \frac{1}{z} \cdot \frac{1}{1+(1/z)}$$

in negative powers of $z$ that is valid when $1 < |z| < \infty$.

*Ans.* $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{z^n}$.

**4.** Give two Laurent series expansions in powers of $z$ for the function

$$f(z) = \frac{1}{z^2(1-z)},$$

and specify the regions in which those expansions are valid.

*Ans.* $\displaystyle\sum_{n=0}^{\infty} z^n + \frac{1}{z} + \frac{1}{z^2} \quad (0 < |z| < 1); \qquad -\sum_{n=3}^{\infty} \frac{1}{z^n} \quad (1 < |z| < \infty)$.

**5.** Represent the function

$$f(z) = \frac{z+1}{z-1}$$

*(a)* by its Maclaurin series, and state where the representation is valid;
*(b)* by its Laurent series in the domain $1 < |z| < \infty$.

*Ans.* *(a)* $\displaystyle -1 - 2\sum_{n=1}^{\infty} z^n \quad (|z| < 1)$; *(b)* $\displaystyle 1 + 2\sum_{n=1}^{\infty} \frac{1}{z^n}$.

**6.** Show that when $0 < |z-1| < 2$,

$$\frac{z}{(z-1)(z-3)} = -3\sum_{n=0}^{\infty} \frac{(z-1)^n}{2^{n+2}} - \frac{1}{2(z-1)}.$$

**7.** Write the two Laurent series in powers of $z$ that represent the function

$$f(z) = \frac{1}{z(1+z^2)}$$

in certain domains, and specify those domains.

*Ans.* $\displaystyle\sum_{n=0}^{\infty} (-1)^{n+1} z^{2n+1} + \frac{1}{z} \quad (0 < |z| < 1); \qquad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{z^{2n+1}} \quad (1 < |z| < \infty)$.

**8.** *(a)* Let $a$ denote a real number, where $-1 < a < 1$, and derive the Laurent series representation

$$\frac{a}{z-a} = \sum_{n=1}^{\infty} \frac{a^n}{z^n} \qquad (|a| < |z| < \infty).$$

*(b)* After writing $z = e^{i\theta}$ in the equation obtained in part *(a)*, equate real parts and then imaginary parts on each side of the result to derive the summation formulas

$$\sum_{n=1}^{\infty} a^n \cos n\theta = \frac{a\cos\theta - a^2}{1 - 2a\cos\theta + a^2} \quad \text{and} \quad \sum_{n=1}^{\infty} a^n \sin n\theta = \frac{a\sin\theta}{1 - 2a\cos\theta + a^2},$$

where $-1 < a < 1$. (Compare with Exercise 4, Sec. 56.)

**9.** Suppose that a series

$$\sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

converges to an analytic function $X(z)$ in some annulus $R_1 < |z| < R_2$. That sum $X(z)$ is called the *z-transform* of $x[n]$ $(n = 0, \pm 1, \pm 2, \ldots)$.* Use expression (5), Sec. 60, for the coefficients in a Laurent series to show that if the annulus contains the unit circle $|z| = 1$, then the *inverse* $z$-transform of $X(z)$ can be written

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{i\theta})e^{in\theta}\, d\theta \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

**10.** *(a)* Let $z$ be any complex number, and let $C$ denote the unit circle

$$w = e^{i\phi} \qquad (-\pi \le \phi \le \pi)$$

in the $w$ plane. Then use that contour in expression (5), Sec. 60, for the coefficients in a Laurent series, adapted to such series about the origin in the $w$ plane, to show that

$$\exp\left[\frac{z}{2}\left(w - \frac{1}{w}\right)\right] = \sum_{n=-\infty}^{\infty} J_n(z)w^n \qquad (0 < |w| < \infty)$$

where

$$J_n(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp[-i(n\phi - z\sin\phi)]\, d\phi \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

*(b)* With the aid of Exercise 5, Sec. 38, regarding certain definite integrals of even and odd complex-valued functions of a real variable, show that the coefficients in part *(a)* here can be written†

$$J_n(z) = \frac{1}{\pi}\int_{0}^{\pi} \cos(n\phi - z\sin\phi)\, d\phi \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

---

*The $z$-transform arises in studies of discrete-time linear systems. See, for instance, the book by Oppenheim, Schafer, and Buck that is listed in Appendix 1.

†These coefficients $J_n(z)$ are called *Bessel functions* of the first kind. They play a prominent role in certain areas of applied mathematics. See, for example, the authors' "Fourier Series and Boundary Value Problems," 7th ed., Chap. 9, 2008.

**11.** *(a)* Let $f(z)$ denote a function which is analytic in some annular domain about the origin that includes the unit circle $z = e^{i\phi}$ $(-\pi \le \phi \le \pi)$. By taking that circle as the path of integration in expressions (2) and (3), Sec. 60, for the coefficients $a_n$ and $b_n$ in a Laurent series in powers of $z$, show that

$$f(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\phi})\,d\phi + \frac{1}{2\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} f(e^{i\phi})\left[\left(\frac{z}{e^{i\phi}}\right)^n + \left(\frac{e^{i\phi}}{z}\right)^n\right]d\phi$$

when $z$ is any point in the annular domain.

*(b)* Write $u(\theta) = \text{Re}[f(e^{i\theta})]$ and show how it follows from the expansion in part *(a)* that

$$u(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(\phi)\,d\phi + \frac{1}{\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} u(\phi)\cos[n(\theta-\phi)]\,d\phi.$$

This is one form of the *Fourier series* expansion of the real-valued function $u(\theta)$ on the interval $-\pi \le \theta \le \pi$. The restriction on $u(\theta)$ is more severe than is necessary in order for it to be represented by a Fourier series.*

## 63. ABSOLUTE AND UNIFORM CONVERGENCE OF POWER SERIES

This section and the three following it are devoted mainly to various properties of power series. A reader who wishes to simply accept the theorems and the corollary in these sections can easily skip the proofs in order to reach Sec. 67 more quickly.

We recall from Sec. 56 that a series of complex numbers converges *absolutely* if the series of absolute values of those numbers converges. The following theorem concerns the absolute convergence of power series.

***Theorem 1.*** *If a power series*

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n \tag{1}$$

*converges when* $z = z_1$ $(z_1 \ne z_0)$, *then it is absolutely convergent at each point* $z$ *in the open disk* $|z - z_0| < R_1$ *where* $R_1 = |z_1 - z_0|$ (Fig. 79).

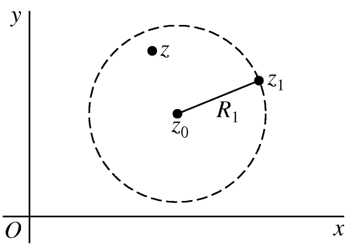


**FIGURE 79**

*For other sufficient conditions, see Secs. 12 and 13 of the book cited in the footnote to Exercise 10.

That is,

$$\frac{1}{1+z^2/3!+z^4/5!+\cdots}=1-\frac{1}{3!}z^2+\left[\frac{1}{(3!)^2}-\frac{1}{5!}\right]z^4+\cdots,$$

or

$$\frac{1}{1+z^2/3!+z^4/5!+\cdots}=1-\frac{1}{6}z^2+\frac{7}{360}z^4+\cdots \qquad (|z|<\pi). \tag{8}$$

Hence

$$\frac{1}{z^2\sinh z}=\frac{1}{z^3}-\frac{1}{6}\cdot\frac{1}{z}+\frac{7}{360}z+\cdots \qquad (0<|z|<\pi). \tag{9}$$

Although we have given only the first three nonzero terms of this Laurent series, any number of terms can, of course, be found by continuing the division.

## EXERCISES

**1.** Use multiplication of series to show that

$$\frac{e^z}{z(z^2+1)}=\frac{1}{z}+1-\frac{1}{2}z-\frac{5}{6}z^2+\cdots \qquad (0<|z|<1).$$

**2.** By writing $\csc z = 1/\sin z$ and then using division, show that

$$\csc z=\frac{1}{z}+\frac{1}{3!}z+\left[\frac{1}{(3!)^2}-\frac{1}{5!}\right]z^3+\cdots \qquad (0<|z|<\pi).$$

**3.** Use division to obtain the Laurent series representation

$$\frac{1}{e^z-1}=\frac{1}{z}-\frac{1}{2}+\frac{1}{12}z-\frac{1}{720}z^3+\cdots \qquad (0<|z|<2\pi).$$

**4.** Use the expansion

$$\frac{1}{z^2\sinh z}=\frac{1}{z^3}-\frac{1}{6}\cdot\frac{1}{z}+\frac{7}{360}z+\cdots \qquad (0<|z|<\pi)$$

in Example 2, Sec. 67, and the method illustrated in Example 1, Sec. 62, to show that

$$\int_C \frac{dz}{z^2\sinh z}=-\frac{\pi i}{3},$$

when $C$ is the positively oriented unit circle $|z|=1$.

**5.** Follow these steps, which illustrate an alternative to straightforward division, to obtain representation (8) in Example 2, Sec. 67.

*(a)* Write

$$\frac{1}{1+z^2/3!+z^4/5!+\cdots}=d_0+d_1z+d_2z^2+d_3z^3+d_4z^4+\cdots,$$

where the coefficients in the power series on the right are to be determined by multiplying the two series in the equation

$$1 = \left(1 + \frac{1}{3!}z^2 + \frac{1}{5!}z^4 + \cdots\right)(d_0 + d_1 z + d_2 z^2 + d_3 z^3 + d_4 z^4 + \cdots).$$

Perform this multiplication to show that

$$(d_0 - 1) + d_1 z + \left(d_2 + \frac{1}{3!}d_0\right)z^2 + \left(d_3 + \frac{1}{3!}d_1\right)z^3 + \left(d_4 + \frac{1}{3!}d_2 + \frac{1}{5!}d_0\right)z^4 + \cdots = 0$$

when $|z| < \pi$.

*(b)* By setting the coefficients in the last series in part *(a)* equal to zero, find the values of $d_0, d_1, d_2, d_3$, and $d_4$. With these values, the first equation in part *(a)* becomes equation (8), Sec. 67.

**6.** Use mathematical induction to establish Leibniz' rule (Sec. 67)

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)} \qquad (n = 1, 2, \ldots)$$

for the $n^{th}$ derivative of the product of two differentiable functions $f(z)$ and $g(z)$.

*Suggestion*: Note that the rule is valid when $n = 1$. Then, assuming that it is valid when $n = m$ where $m$ is any positive integer, show that

$$\begin{aligned}(fg)^{(m+1)} &= (fg')^{(m)} + (f'g)^{(m)} \\ &= fg^{(m+1)} + \sum_{k=1}^{m}\left[\binom{m}{k} + \binom{m}{k-1}\right] f^{(k)} g^{(m+1-k)} + f^{(m+1)}g.\end{aligned}$$

Finally, with the aid of the identify

$$\binom{m}{k} + \binom{m}{k-1} = \binom{m+1}{k}$$

that was used in Exercise 8, Sec. 3, show that

$$\begin{aligned}(fg)^{(m+1)} &= fg^{(m+1)} + \sum_{k=1}^{m}\binom{m+1}{k} f^{(k)} g^{(m+1-k)} + f^{(m+1)}g \\ &= \sum_{k=0}^{m+1}\binom{m+1}{k} f^{(k)} g^{(m+1-k)}.\end{aligned}$$

**7.** Let $f(z)$ be an entire function that is represented by a series of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \qquad (|z| < \infty).$$

*(a)* By differentiating the composite function $g(z) = f[f(z)]$ successively, find the first three nonzero terms in the Maclaurin series for $g(z)$ and thus show that

$$f[f(z)] = z + 2\,a_2 z^2 + 2\,(a_2^2 + a_3)z^3 + \cdots \qquad (|z| < \infty).$$

*(b)* Obtain the result in part *(a)* in a *formal* manner by writing

$$f[f(z)] = f(z) + a_2[f(z)]^2 + a_3[f(z)]^3 + \cdots,$$

replacing $f(z)$ on the right-hand side here by its series representation, and then collecting terms in like powers of $z$.

*(c)* By applying the result in part *(a)* to the function $f(z) = \sin z$, show that

$$\sin(\sin z) = z - \frac{1}{3}z^3 + \cdots \qquad (|z| < \infty).$$

**8.** The *Euler numbers* are the numbers $E_n$ $(n = 0, 1, 2, \ldots)$ in the Maclaurin series representation

$$\frac{1}{\cosh z} = \sum_{n=0}^{\infty} \frac{E_n}{n!} z^n \qquad (|z| < \pi/2).$$

Point out why this representation is valid in the indicated disk and why

$$E_{2n+1} = 0 \qquad (n = 0, 1, 2, \ldots).$$

Then show that

$$E_0 = 1, \quad E_2 = -1, \quad E_4 = 5, \quad \text{and} \quad E_6 = -61.$$