which f is analytic, then the value of the integral of f over C_1 never changes. To verify the corollary, we need only write equation (2) as

$$\int_{C_2} f(z)\,dz + \int_{-C_1} f(z)\,dz = 0$$

and apply the theroem.

EXAMPLE. When C is any positively oriented simple closed contour surrounding the origin, the corollary can be used to show that

$$\int_C \frac{dz}{z} = 2\pi i.$$

This is done by constructing a positively oriented circle C_0 with center at the origin and radius so small that C_0 lies entirely inside C (Fig. 62). Since (see Example 2, Sec. 42)

$$\int_{C_0} \frac{dz}{z} = 2\pi i$$

and since $1/z$ is analytic everywhere except at $z = 0$, the desired result follows.

Note that the radius of C_0 could equally well have been so large that C lies entirely inside C_0.

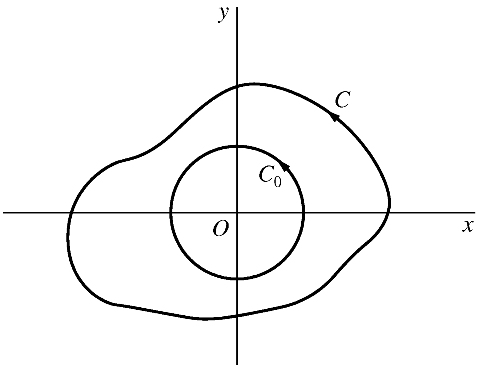

FIGURE 62

EXERCISES

1. Apply the Cauchy–Goursat theorem to show that

$$\int_C f(z)\,dz = 0$$

when the contour C is the unit circle $|z| = 1$, in either direction, and when

(a) $f(z) = \dfrac{z^2}{z-3}$; *(b)* $f(z) = z\,e^{-z}$; *(c)* $f(z) = \dfrac{1}{z^2+2z+2}$;

(d) $f(z) = \operatorname{sech} z$; *(e)* $f(z) = \tan z$; *(f)* $f(z) = \operatorname{Log}(z+2)$.

2. Let C_1 denote the positively oriented boundary of the square whose sides lie along the lines $x = \pm 1$, $y = \pm 1$ and let C_2 be the positively oriented circle $|z| = 4$ (Fig. 63). With the aid of the corollary in Sec. 49, point out why

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

when

(a) $f(z) = \dfrac{1}{3z^2+1}$; *(b)* $f(z) = \dfrac{z+2}{\sin(z/2)}$; *(c)* $f(z) = \dfrac{z}{1-e^z}$.

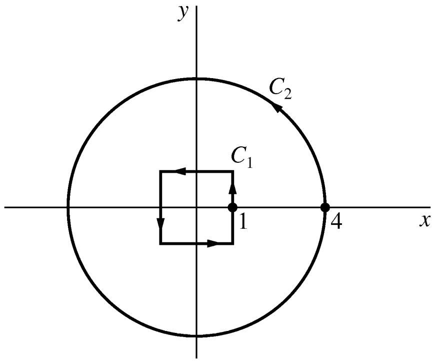

FIGURE 63

3. If C_0 denotes a positively oriented circle $|z - z_0| = R$, then

$$\int_{C_0} (z-z_0)^{n-1}\,dz = \begin{cases} 0 & \text{when } n = \pm 1, \pm 2, \ldots, \\ 2\pi i & \text{when } n = 0, \end{cases}$$

according to Exercise 10*(b)*, Sec. 42. Use that result and the corollary in Sec. 49 to show that if C is the boundary of the rectangle $0 \le x \le 3$, $0 \le y \le 2$, described in the positive sense, then

$$\int_{C} (z-2-i)^{n-1}\,dz = \begin{cases} 0 & \text{when } n = \pm 1, \pm 2, \ldots, \\ 2\pi i & \text{when } n = 0. \end{cases}$$

4. Use the following method to derive the integration formula

$$\int_0^\infty e^{-x^2} \cos 2bx\,dx = \frac{\sqrt{\pi}}{2} e^{-b^2} \qquad (b > 0).$$

(a) Show that the sum of the integrals of e^{-z^2} along the lower and upper horizontal legs of the rectangular path in Fig. 64 can be written

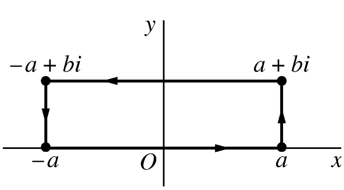

FIGURE 64

$$2\int_0^a e^{-x^2}dx - 2e^{b^2}\int_0^a e^{-x^2}\cos 2bx\,dx$$

and that the sum of the integrals along the vertical legs on the right and left can be written

$$ie^{-a^2}\int_0^b e^{y^2}e^{-i2ay}dy - ie^{-a^2}\int_0^b e^{y^2}e^{i2ay}dy.$$

Thus, with the aid of the Cauchy–Goursat theorem, show that

$$\int_0^a e^{-x^2}\cos 2bx\,dx = e^{-b^2}\int_0^a e^{-x^2}dx + e^{-(a^2+b^2)}\int_0^b e^{y^2}\sin 2ay\,dy.$$

(b) By accepting the fact that*

$$\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$$

and observing that

$$\left|\int_0^b e^{y^2}\sin 2ay\,dy\right| \le \int_0^b e^{y^2}dy,$$

obtain the desired integration formula by letting a tend to infinity in the equation at the end of part *(a)*.

5. According to Exercise 6, Sec. 39, the path C_1 from the origin to the point $z = 1$ along the graph of the function defined by means of the equations

$$y(x) = \begin{cases} x^3\sin(\pi/x) & \text{when } 0 < x \le 1, \\ 0 & \text{when } x = 0 \end{cases}$$

is a smooth arc that intersects the real axis an infinite number of times. Let C_2 denote the line segment along the real axis from $z = 1$ back to the origin, and let C_3 denote any smooth arc from the origin to $z = 1$ that does not intersect itself and has only its end points in common with the arcs C_1 and C_2 (Fig. 65). Apply the Cauchy–Goursat theorem to show that if a function f is entire, then

$$\int_{C_1} f(z)\,dz = \int_{C_3} f(z)\,dz \quad \text{and} \quad \int_{C_2} f(z)\,dz = -\int_{C_3} f(z)\,dz.$$

Conclude that even though the closed contour $C = C_1 + C_2$ intersects itself an infinite number of times,

$$\int_C f(z)\,dz = 0.$$

*The usual way to evaluate this integral is by writing its square as

$$\int_0^\infty e^{-x^2}dx\int_0^\infty e^{-y^2}dy = \int_0^\infty\int_0^\infty e^{-(x^2+y^2)}dx\,dy$$

and then evaluating this iterated integral by changing to polar coordinates. Details are given in, for example, A. E. Taylor and W. R. Mann, "Advanced Calculus," 3d ed., pp. 680–681, 1983.

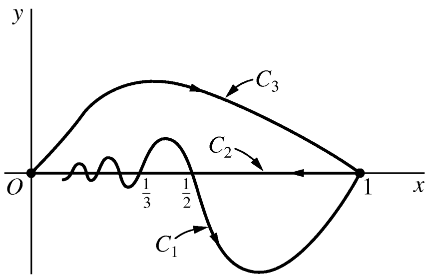

FIGURE 65

6. Let C denote the positively oriented boundary of the half disk $0 \le r \le 1, 0 \le \theta \le \pi$, and let $f(z)$ be a continuous function defined on that half disk by writing $f(0) = 0$ and using the branch

$$f(z) = \sqrt{r}e^{i\theta/2} \qquad \left(r > 0, -\frac{\pi}{2} < \theta < \frac{3\pi}{2}\right)$$

of the multiple-valued function $z^{1/2}$. Show that

$$\int_C f(z)\,dz = 0$$

by evaluating separately the integrals of $f(z)$ over the semicircle and the two radii which make up C. Why does the Cauchy–Goursat theorem not apply here?

7. Show that if C is a positively oriented simple closed contour, then the area of the region enclosed by C can be written

$$\frac{1}{2i}\int_C \overline{z}\,dz.$$

Suggestion: Note that expression (4), Sec. 46, can be used here even though the function $f(z) = \overline{z}$ is not analytic anywhere [see Example 2, Sec. 19].

8. *Nested Intervals.* An infinite sequence of closed intervals $a_n \le x \le b_n$ $(n = 0, 1, 2, \ldots)$ is formed in the following way. The interval $a_1 \le x \le b_1$ is either the left-hand or right-hand half of the first interval $a_0 \le x \le b_0$, and the interval $a_2 \le x \le b_2$ is then one of the two halves of $a_1 \le x \le b_1$, etc. Prove that there is a point x_0 which belongs to every one of the closed intervals $a_n \le x \le b_n$.

Suggestion: Note that the left-hand end points a_n represent a bounded nondecreasing sequence of numbers, since $a_0 \le a_n \le a_{n+1} < b_0$; hence they have a limit A as n tends to infinity. Show that the end points b_n also have a limit B. Then show that $A = B$, and write $x_0 = A = B$.

9. *Nested Squares.* A square $\sigma_0 : a_0 \le x \le b_0, c_0 \le y \le d_0$ is divided into four equal squares by line segments parallel to the coordinate axes. One of those four smaller squares $\sigma_1 : a_1 \le x \le b_1, c_1 \le y \le d_1$ is selected according to some rule. It, in turn, is divided into four equal squares one of which, called σ_2, is selected, etc. (see Sec. 47). Prove that there is a point (x_0, y_0) which belongs to each of the closed regions of the infinite sequence $\sigma_0, \sigma_1, \sigma_2, \ldots$.

Suggestion: Apply the result in Exercise 8 to each of the sequences of closed intervals $a_n \le x \le b_n$ and $c_n \le y \le d_n$ $(n = 0, 1, 2, \ldots)$.

Theorem 3. *Suppose that a function f is analytic inside and on a positively oriented circle C_R, centered at z_0 and with radius R* (Fig. 69). *If M_R denotes the maximum value of $|f(z)|$ on C_R, then*

$$|f^{(n)}(z_0)| \le \frac{n!M_R}{R^n} \quad (n = 1, 2, \ldots). \tag{2}$$

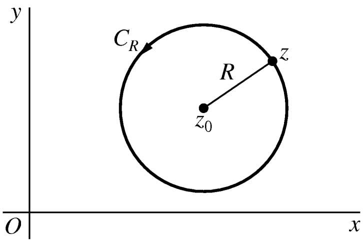

FIGURE 69

Inequality (2) is called *Cauchy's inequality* and is an immediate consequence of the expression

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_{C_R} \frac{f(z)\,dz}{(z-z_0)^{n+1}} \quad (n = 1, 2, \ldots),$$

which is a slightly different form of equation (6), Sec. 51, when n is a positive integer. We need only apply the theorem in Sec. 43, which gives upper bounds for the moduli of the values of contour integrals, to see that

$$|f^{(n)}(z_0)| \le \frac{n!}{2\pi}\cdot\frac{M_R}{R^{n+1}}2\pi R \quad (n = 1, 2, \ldots),$$

where M_R is as in the statement of Theorem 3. This inequality is, of course, the same as inequality (2).

EXERCISES

1. Let C denote the positively oriented boundary of the square whose sides lie along the lines $x = \pm 2$ and $y = \pm 2$. Evaluate each of these integrals:

(a) $\displaystyle\int_C \frac{e^{-z}\,dz}{z-(\pi i/2)}$; *(b)* $\displaystyle\int_C \frac{\cos z}{z(z^2+8)}\,dz$; *(c)* $\displaystyle\int_C \frac{z\,dz}{2z+1}$;

(d) $\displaystyle\int_C \frac{\cosh z}{z^4}\,dz$; *(e)* $\displaystyle\int_C \frac{\tan(z/2)}{(z-x_0)^2}\,dz \quad (-2 < x_0 < 2)$.

Ans. (a) 2π; *(b)* $\pi i/4$; *(c)* $-\pi i/2$; *(d)* 0; *(e)* $i\pi\sec^2(x_0/2)$.

2. Find the value of the integral of $g(z)$ around the circle $|z - i| = 2$ in the positive sense when

(a) $g(z) = \dfrac{1}{z^2+4}$; *(b)* $g(z) = \dfrac{1}{(z^2+4)^2}$.

Ans. (a) $\pi/2$; *(b)* $\pi/16$.

3. Let C be the circle $|z| = 3$, described in the positive sense. Show that if

$$g(z) = \int_C \frac{2s^2 - s - 2}{s - z}\, ds \qquad (|z| \neq 3),$$

then $g(2) = 8\pi i$. What is the value of $g(z)$ when $|z| > 3$?

4. Let C be any simple closed contour, described in the positive sense in the z plane, and write

$$g(z) = \int_C \frac{s^3 + 2s}{(s - z)^3}\, ds.$$

Show that $g(z) = 6\pi i z$ when z is inside C and that $g(z) = 0$ when z is outside.

5. Show that if f is analytic within and on a simple closed contour C and z_0 is not on C, then

$$\int_C \frac{f'(z)\, dz}{z - z_0} = \int_C \frac{f(z)\, dz}{(z - z_0)^2}.$$

6. Let f denote a function that is *continuous* on a simple closed contour C. Following a procedure used in Sec. 51, prove that the function

$$g(z) = \frac{1}{2\pi i} \int_C \frac{f(s)\, ds}{s - z}$$

is *analytic* at each point z interior to C and that

$$g'(z) = \frac{1}{2\pi i} \int_C \frac{f(s)\, ds}{(s - z)^2}$$

at such a point.

7. Let C be the unit circle $z = e^{i\theta}\,(-\pi \le \theta \le \pi)$. First show that for any real constant a,

$$\int_C \frac{e^{az}}{z}\, dz = 2\pi i.$$

Then write this integral in terms of θ to derive the integration formula

$$\int_0^\pi e^{a\cos\theta} \cos(a \sin\theta)\, d\theta = \pi.$$

8. *(a)* With the aid of the binomial formula (Sec. 3), show that for each value of n, the function

$$P_n(z) = \frac{1}{n!\,2^n} \frac{d^n}{dz^n}(z^2 - 1)^n \qquad (n = 0, 1, 2, \ldots)$$

is a polynomial of degree n.*

*These are Legendre polynomials, which appear in Exercise 7, Sec. 43, when $z = x$. See the footnote to that exercise.

(b) Let C denote any positively oriented simple closed contour surrounding a fixed point z. With the aid of the integral representation (5), Sec. 51, for the nth derivative of a function, show that the polynomials in part *(a)* can be expressed in the form

$$P_n(z) = \frac{1}{2^{n+1}\pi i} \int_C \frac{(s^2-1)^n}{(s-z)^{n+1}}\, ds \qquad (n = 0, 1, 2, \ldots).$$

(c) Point out how the integrand in the representation for $P_n(z)$ in part *(b)* can be written $(s+1)^n/(s-1)$ if $z = 1$. Then apply the Cauchy integral formula to show that

$$P_n(1) = 1 \qquad (n = 0, 1, 2, \ldots).$$

Similarly, show that

$$P_n(-1) = (-1)^n \qquad (n = 0, 1, 2, \ldots).$$

9. Follow these steps below to verify the expression

$$f''(z) = \frac{1}{\pi i} \int_C \frac{f(s)\, ds}{(s-z)^3}$$

in Sec. 51.

(a) Use expression (2) in Sec. 51 for $f'(z)$ to show that

$$\frac{f'(z+\Delta z) - f'(z)}{\Delta z} - \frac{1}{\pi i} \int_C \frac{f(s)\, ds}{(s-z)^3} = \frac{1}{2\pi i} \int_C \frac{3(s-z)\Delta z - 2(\Delta z)^2}{(s-z-\Delta z)^2 (s-z)^3} f(s)\, ds.$$

(b) Let D and d denote the largest and smallest distances, respectively, from z to points on C. Also, let M be the maximum value of $|f(s)|$ on C and L the length of C. With the aid of the triangle inequality and by referring to the derivation of expression (2) in Sec. 51 for $f'(z)$, show that when $0 < |\Delta z| < d$, the value of the integral on the right-hand side in part *(a)* is bounded from above by

$$\frac{(3D|\Delta z| + 2|\Delta z|^2)M}{(d - |\Delta z|)^2 d^3} L.$$

(c) Use the results in parts *(a)* and *(b)* to obtain the desired expression for $f''(z)$.

10. Let f be an entire function such that $|f(z)| \le A|z|$ for all z, where A is a fixed positive number. Show that $f(z) = a_1 z$, where a_1 is a complex constant.

Suggestion: Use Cauchy's inequality (Sec. 52) to show that the second derivative $f''(z)$ is zero everywhere in the plane. Note that the constant M_R in Cauchy's inequality is less than or equal to $A(|z_0| + R)$.

53. LIOUVILLE'S THEOREM AND THE FUNDAMENTAL THEOREM OF ALGEBRA

Cauchy's inequality in Theorem 3 of Sec. 52 can be used to show that no entire function except a constant is bounded in the complex plane. Our first theorem here,