$a < t < b$. This expression for $\mathbf{T}$ is the one learned in calculus when $z(t)$ is interpreted as a radius vector. Such an arc is said to be *smooth*. In referring to a smooth arc $z = z(t)$ $(a \le t \le b)$, then, we agree that the derivative $z'(t)$ is continuous on the closed interval $a \le t \le b$ and nonzero throughout the open interval $a < t < b$.

A *contour*, or piecewise smooth arc, is an arc consisting of a finite number of smooth arcs joined end to end. Hence if equation (2) represents a contour, $z(t)$ is continuous, whereas its derivative $z'(t)$ is piecewise continuous. The polygonal line (4) is, for example, a contour. When only the initial and final values of $z(t)$ are the same, a contour C is called a *simple closed contour*. Examples are the circles (5) and (6), as well as the boundary of a triangle or a rectangle taken in a specific direction. The length of a contour or a simple closed contour is the sum of the lengths of the smooth arcs that make up the contour.

The points on any simple closed curve or simple closed contour C are boundary points of two distinct domains, one of which is the interior of C and is bounded. The other, which is the exterior of C, is unbounded. It will be convenient to accept this statement, known as the *Jordan curve theorem*, as geometrically evident; the proof is not easy.*

EXERCISES

1. Show that if $w(t) = u(t) + iv(t)$ is continuous on an interval $a \le t \le b$, then

(a) $\displaystyle\int_{-b}^{-a} w(-t)\,dt = \int_a^b w(\tau)\,d\tau;$

(b) $\displaystyle\int_a^b w(t)\,dt = \int_\alpha^\beta w[\phi(\tau)]\phi'(\tau)\,d\tau,$ where $\phi(\tau)$ is the function in equation (9), Sec. 39.

Suggestion: These identities can be obtained by noting that they are valid for *real-valued* functions of t.

2. Let C denote the right-hand half of the circle $|z| = 2$, in the counterclockwise direction, and note that two parametric representations for C are

$$z = z(\theta) = 2\,e^{i\theta} \qquad \left(-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right)$$

and

$$z = Z(y) = \sqrt{4 - y^2} + iy \qquad (-2 \le y \le 2).$$

Verify that $Z(y) = z[\phi(y)]$, where

$$\phi(y) = \arctan\frac{y}{\sqrt{4 - y^2}} \qquad \left(-\frac{\pi}{2} < \arctan t < \frac{\pi}{2}\right).$$

*See pp. 115–116 of the book by Newman or Sec. 13 of the one by Thron, both of which are cited in Appendix 1. The special case in which C is a simple closed polygon is proved on pp. 281–285 of Vol. 1 of the work by Hille, also cited in Appendix 1.

Also, show that this function ϕ has a positive derivative, as required in the conditions following equation (9), Sec. 39.

3. Derive the equation of the line through the points (α, a) and (β, b) in the τt plane that are shown in Fig. 37. Then use it to find the linear function $\phi(\tau)$ which can be used in equation (9), Sec. 39, to transform representation (2) in that section into representation (10) there.

Ans. $\phi(\tau) = \dfrac{b-a}{\beta-\alpha}\tau + \dfrac{a\beta - b\alpha}{\beta-\alpha}$.

4. Verify expression (14), Sec. 39, for the derivative of $Z(\tau) = z[\phi(\tau)]$.

Suggestion: Write $Z(\tau) = x[\phi(\tau)] + iy[\phi(\tau)]$ and apply the chain rule for real-valued functions of a real variable.

5. Suppose that a function $f(z)$ is analytic at a point $z_0 = z(t_0)$ lying on a smooth arc $z = z(t)$ $(a \le t \le b)$. Show that if $w(t) = f[z(t)]$, then

$$w'(t) = f'[z(t)]z'(t)$$

when $t = t_0$.

Suggestion: Write $f(z) = u(x, y) + iv(x, y)$ and $z(t) = x(t) + iy(t)$, so that

$$w(t) = u[x(t), y(t)] + iv[x(t), y(t)].$$

Then apply the chain rule in calculus for functions of two real variables to write

$$w' = (u_x x' + u_y y') + i(v_x x' + v_y y'),$$

and use the Cauchy–Riemann equations.

6. Let $y(x)$ be a real-valued function defined on the interval $0 \le x \le 1$ by means of the equations

$$y(x) = \begin{cases} x^3 \sin(\pi/x) & \text{when } 0 < x \le 1, \\ 0 & \text{when } x = 0. \end{cases}$$

(a) Show that the equation

$$z = x + iy(x) \qquad (0 \le x \le 1)$$

represents an arc C that intersects the real axis at the points $z = 1/n$ $(n = 1, 2, \ldots)$ and $z = 0$, as shown in Fig. 38.

(b) Verify that the arc C in part *(a)* is, in fact, a *smooth* arc.

Suggestion: To establish the continuity of $y(x)$ at $x = 0$, observe that

$$0 \le \left| x^3 \sin\left(\frac{\pi}{x}\right) \right| \le x^3$$

when $x > 0$. A similar remark applies in finding $y'(0)$ and showing that $y'(x)$ is continuous at $x = 0$.

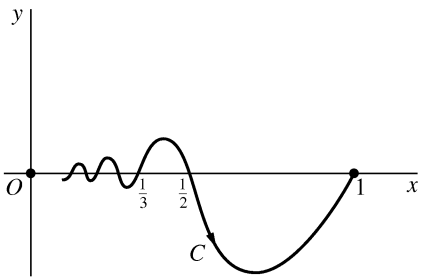

FIGURE 38

40. CONTOUR INTEGRALS

We turn now to integrals of complex-valued functions f of the complex variable z. Such an integral is defined in terms of the values $f(z)$ along a given contour C, extending from a point $z = z_1$ to a point $z = z_2$ in the complex plane. It is, therefore, a line integral; and its value depends, in general, on the contour C as well as on the function f. It is written

$$\int_C f(z)\, dz \quad \text{or} \quad \int_{z_1}^{z_2} f(z)\, dz,$$

the latter notation often being used when the value of the integral is independent of the choice of the contour taken between two fixed end points. While the integral may be defined directly as the limit of a sum, we choose to define it in terms of a definite integral of the type introduced in Sec. 38.

Suppose that the equation

$$z = z(t) \quad (a \le t \le b) \tag{1}$$

represents a contour C, extending from a point $z_1 = z(a)$ to a point $z_2 = z(b)$. We assume that $f[z(t)]$ is *piecewise continuous* (Sec. 38) on the interval $a \le t \le b$ and refer to the function $f(z)$ as being piecewise continuous on C. We then define the line integral, or *contour integral*, of f along C in terms of the parameter t:

$$\int_C f(z)\, dz = \int_a^b f[z(t)]z'(t)\, dt. \tag{2}$$

Note that since C is a contour, $z'(t)$ is also piecewise continuous on $a \le t \le b$; and so the existence of integral (2) is ensured.

The value of a contour integral is invariant under a change in the representation of its contour when the change is of the type (11), Sec. 39. This can be seen by following the same general procedure that was used in Sec. 39 to show the invariance of arc length.

Note that if a is a nonzero integer n, this result tells us that

$$\int_C z^{n-1}\,dz = 0 \quad (n = \pm 1, \pm 2, \ldots). \tag{5}$$

If a is allowed to be zero, we have

$$\int_C \frac{dz}{z} = \int_{-\pi}^{\pi} \frac{1}{Re^{i\theta}} iRe^{i\theta}\,d\theta = i\int_{-\pi}^{\pi} d\theta = 2\pi i. \tag{6}$$

EXERCISES

For the functions f and contours C in Exercises 1 through 7, use parametric representations for C, or legs of C, to evaluate

$$\int_C f(z)\,dz.$$

1. $f(z) = (z+2)/z$ and C is

(a) the semicircle $z = 2e^{i\theta}$ $(0 \le \theta \le \pi)$;

(b) the semicircle $z = 2e^{i\theta}$ $(\pi \le \theta \le 2\pi)$;

(c) the circle $z = 2e^{i\theta}$ $(0 \le \theta \le 2\pi)$.

Ans. (a) $-4 + 2\pi i$; *(b)* $4 + 2\pi i$; *(c)* $4\pi i$.

2. $f(z) = z - 1$ and C is the arc from $z = 0$ to $z = 2$ consisting of

(a) the semicircle $z = 1 + e^{i\theta}$ $(\pi \le \theta \le 2\pi)$;

(b) the segment $z = x$ $(0 \le x \le 2)$ of the real axis.

Ans. (a) 0; *(b)* 0.

3. $f(z) = \pi \exp(\pi \bar{z})$ and C is the boundary of the square with vertices at the points 0, 1, $1 + i$, and i, the orientation of C being in the counterclockwise direction.

Ans. $4(e^{\pi} - 1)$.

4. $f(z)$ is defined by means of the equations

$$f(z) = \begin{cases} 1 & \text{when } y < 0, \\ 4y & \text{when } y > 0, \end{cases}$$

and C is the arc from $z = -1 - i$ to $z = 1 + i$ along the curve $y = x^3$.

Ans. $2 + 3i$.

5. $f(z) = 1$ and C is an arbitrary contour from any fixed point z_1 to any fixed point z_2 in the z plane.

Ans. $z_2 - z_1$.

6. $f(z)$ is the branch

$$z^{-1+i} = \exp[(-1+i)\log z] \qquad (|z| > 0, 0 < \arg z < 2\pi)$$

of the indicated power function, and C is the unit circle $z = e^{i\theta}$ $(0 \le \theta \le 2\pi)$.

Ans. $i(1 - e^{-2\pi})$.

7. $f(z)$ is the principal branch

$$z^i = \exp(i\text{Log } z) \quad (|z| > 0, -\pi < \text{ Arg } z < \pi)$$

of this power function, and C is the semicircle $z = e^{i\theta}$ $(0 \le \theta \le \pi)$.

Ans. $-\dfrac{1+e^{-\pi}}{2}(1-i)$.

8. With the aid of the result in Exercise 3, Sec. 38, evaluate the integral

$$\int_C z^m \bar{z}^n dz,$$

where m and n are integers and C is the unit circle $|z| = 1$, taken counterclockwise.

9. Evaluate the integral I in Example 1, Sec. 41, using this representation for C:

$$z = \sqrt{4-y^2} + iy \qquad (-2 \le y \le 2).$$

(See Exercise 2, Sec. 39.)

10. Let C_0 and C denote the circles

$$z = z_0 + Re^{i\theta} \ (-\pi \le \theta \le \pi) \quad \text{and} \quad z = Re^{i\theta} \ (-\pi \le \theta \le \pi),$$

respectively.

(a) Use these parametric representations to show that

$$\int_{C_0} f(z - z_0)\, dz = \int_C f(z)\, dz$$

when f is piecewise continuous on C.

(b) Apply the result in part *(a)* to integrals (5) and (6) in Sec. 42 to show that

$$\int_{C_0} (z - z_0)^{n-1}\, dz = 0 \ (n = \pm 1, \pm 2, \ldots) \quad \text{and} \quad \int_{C_0} \frac{dz}{z - z_0} = 2\pi i.$$

11. *(a)* Suppose that a function $f(z)$ is continuous on a smooth arc C, which has a parametric representation $z = z(t)$ $(a \le t \le b)$; that is, $f[z(t)]$ is continuous on the interval $a \le t \le b$. Show that if $\phi(\tau)$ $(\alpha \le \tau \le \beta)$ is the function described in Sec. 39, then

$$\int_a^b f[z(t)]z'(t)\, dt = \int_\alpha^\beta f[Z(\tau)]Z'(\tau)\, d\tau$$

where $Z(\tau) = z[\phi(\tau)]$.

(b) Point out how it follows that the identity obtained in part *(a)* remains valid when C is any contour, not necessarily a smooth one, and $f(z)$ is piecewise continuous on C. Thus show that the value of the integral of $f(z)$ along C is the same when the representation $z = Z(\tau)$ $(\alpha \le \tau \le \beta)$ is used, instead of the original one.

Suggestion: In part *(a)*, use the result in Exercise 1*(b)*, Sec. 39, and then refer to expression (14) in that section.

Consequently, at points on C_R,

$$\left|\frac{z^{1/2}}{z^2+1}\right| \le M_R \quad \text{where} \quad M_R = \frac{\sqrt{R}}{R^2-1}.$$

Since the length of C_R is the number $L = \pi R$, it follows from inequality (5) that

$$\left|\int_{C_R} \frac{z^{1/2}}{z^2+1}\, dz\right| \le M_R L.$$

But

$$M_R L = \frac{\pi R\sqrt{R}}{R^2-1} \cdot \frac{1/R^2}{1/R^2} = \frac{\pi/\sqrt{R}}{1-(1/R^2)},$$

and it is clear that the term on the far right here tends to zero as R tends to infinity. Limit (7) is, therefore, established.

EXERCISES

1. Without evaluating the integral, show that

$$\left|\int_C \frac{dz}{z^2-1}\right| \le \frac{\pi}{3}$$

when C is the same arc as the one in Example 1, Sec. 43.

2. Let C denote the line segment from $z = i$ to $z = 1$. By observing that of all the points on that line segment, the midpoint is the closest to the origin, show that

$$\left|\int_C \frac{dz}{z^4}\right| \le 4\sqrt{2}$$

without evaluating the integral.

3. Show that if C is the boundary of the triangle with vertices at the points 0, $3i$, and -4, oriented in the counterclockwise direction (see Fig. 48), then

$$\left|\int_C (e^z - \bar{z})\, dz\right| \le 60.$$

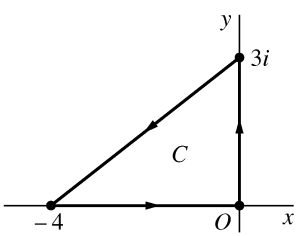

FIGURE 48

4. Let C_R denote the upper half of the circle $|z| = R$ $(R > 2)$, taken in the counterclockwise direction. Show that

$$\left| \int_{C_R} \frac{2z^2 - 1}{z^4 + 5z^2 + 4}\, dz \right| \le \frac{\pi R(2R^2 + 1)}{(R^2 - 1)(R^2 - 4)}.$$

Then, by dividing the numerator and denominator on the right here by R^4, show that the value of the integral tends to zero as R tends to infinity.

5. Let C_R be the circle $|z| = R$ $(R > 1)$, described in the counterclockwise direction. Show that

$$\left| \int_{C_R} \frac{\operatorname{Log} z}{z^2}\, dz \right| < 2\pi \left(\frac{\pi + \ln R}{R} \right),$$

and then use l'Hospital's rule to show that the value of this integral tends to zero as R tends to infinity.

6. Let C_ρ denote a circle $|z| = \rho$ $(0 < \rho < 1)$, oriented in the counterclockwise direction, and suppose that $f(z)$ is analytic in the disk $|z| \le 1$. Show that if $z^{-1/2}$ represents any particular branch of that power of z, then there is a nonnegative constant M, *independent of* ρ, such that

$$\left| \int_{C_\rho} z^{-1/2} f(z)\, dz \right| \le 2\pi M \sqrt{\rho}.$$

Thus show that the value of the integral here approaches 0 as ρ tends to 0.

Suggestion: Note that since $f(z)$ is analytic, and therefore continuous, throughout the disk $|z| \le 1$, it is bounded there (Sec. 18).

7. Apply inequality (1), Sec. 43, to show that for all values of x in the interval $-1 \le x \le 1$, the functions*

$$P_n(x) = \frac{1}{\pi} \int_0^\pi (x + i\sqrt{1 - x^2} \cos\theta)^n\, d\theta \qquad (n = 0, 1, 2, \ldots)$$

satisfy the inequality $|P_n(x)| \le 1$.

8. Let C_N denote the boundary of the square formed by the lines

$$x = \pm \left(N + \frac{1}{2} \right) \pi \quad \text{and} \quad y = \pm \left(N + \frac{1}{2} \right) \pi,$$

where N is a positive integer and the orientation of C_N is counterclockwise.

(a) With the aid of the inequalities

$$|\sin z| \ge |\sin x| \quad \text{and} \quad |\sin z| \ge |\sinh y|,$$

obtained in Exercises 8*(a)* and 9*(a)* of Sec. 34, show that $|\sin z| \ge 1$ on the vertical sides of the square and that $|\sin z| > \sinh(\pi/2)$ on the horizontal sides. Thus show that there is a positive constant A, *independent of* N, such that $|\sin z| \ge A$ for all points z lying on the contour C_N.

*These functions are actually polynomials in x. They are known as *Legendre polynomials* and are important in applied mathematics. See, for example, Chap. 4 of the book by Lebedev that is listed in Appendix 1.

(b) Using the final result in part *(a)*, show that

$$\left| \int_{C_N} \frac{dz}{z^2 \sin z} \right| \leq \frac{16}{(2N+1)\pi A}$$

and hence that the value of this integral tends to zero as N tends to infinity.

44. ANTIDERIVATIVES

Although the value of a contour integral of a function $f(z)$ from a fixed point z_1 to a fixed point z_2 depends, in general, on the path that is taken, there are certain functions whose integrals from z_1 to z_2 have values that are *independent of path.* (Recall Examples 2 and 3 in Sec. 41.) The examples just cited also illustrate the fact that the values of integrals around closed paths are sometimes, but not always, zero. Our next theorem is useful in determining when integration is independent of path and, moreover, when an integral around a closed path has value zero.

The theorem contains an extension of the fundamental theorem of calculus that simplifies the evaluation of many contour integrals. The extension involves the concept on an antiderivative of a continuous function $f(z)$ on a domain D, or a function $F(z)$ such that $F'(z) = f(z)$ for all z in D. Note that an antiderivative is, of necessity, an analytic function. Note, too, that *an antiderivative of a given function $f(z)$ is unique except for an additive constant.* This is because the derivative of the difference $F(z) - G(z)$ of any two such antiderivatives is zero; and, according to the theorem in Sec. 24, an analytic function is constant in a domain D when its derivative is zero throughout D.

Theorem. *Suppose that a function $f(z)$ is continuous on a domain D. If any one of the following statements is true, then so are the others:*

(a) $f(z)$ has an antiderivative $F(z)$ throughout D;

(b) the integrals of $f(z)$ along contours lying entirely in D and extending from any fixed point z_1 to any fixed point z_2 all have the same value, namely

$$\int_{z_1}^{z_2} f(z)\,dz = F(z)\Big]_{z_1}^{z_2} = F(z_2) - F(z_1)$$

where $F(z)$ is the antiderivative in statement (a);

(c) the integrals of $f(z)$ around closed contours lying entirely in D all have value zero.

It should be emphasized that the theorem does *not* claim that any of these statements is true for a given function $f(z)$. It says only that all of them are true or that none of them is true. The next section is devoted to the proof of the theorem and can be easily skipped by a reader who wishes to get on with other important aspects of integration theory. But we include here a number of examples illustrating how the theorem can be used.