EXERCISES

1. Show that

(a) $\text{Log}(-ei) = 1 - \dfrac{\pi}{2}i$; *(b)* $\text{Log}(1-i) = \dfrac{1}{2}\ln 2 - \dfrac{\pi}{4}i$.

2. Show that

(a) $\log e = 1 + 2n\pi i \quad (n = 0, \pm 1, \pm 2, \ldots)$;

(b) $\log i = \left(2n + \dfrac{1}{2}\right)\pi i \quad (n = 0, \pm 1, \pm 2, \ldots)$;

(c) $\log(-1 + \sqrt{3}i) = \ln 2 + 2\left(n + \dfrac{1}{3}\right)\pi i \quad (n = 0, \pm 1, \pm 2, \ldots)$.

3. Show that

(a) $\text{Log}(1+i)^2 = 2\,\text{Log}(1+i)$; *(b)* $\text{Log}(-1+i)^2 \neq 2\,\text{Log}(-1+i)$.

4. Show that

(a) $\log(i^2) = 2\log i$ when $\log z = \ln r + i\theta \left(r > 0, \dfrac{\pi}{4} < \theta < \dfrac{9\pi}{4}\right)$;

(b) $\log(i^2) \neq 2\log i$ when $\log z = \ln r + i\theta \left(r > 0, \dfrac{3\pi}{4} < \theta < \dfrac{11\pi}{4}\right)$.

5. Show that

(a) the set of values of $\log(i^{1/2})$ is

$$\left(n + \frac{1}{4}\right)\pi i \qquad (n = 0, \pm 1, \pm 2, \ldots)$$

and that the same is true of $(1/2)\log i$;

(b) the set of values of $\log(i^2)$ is *not* the same as the set of values of $2\log i$.

6. Given that the branch $\log z = \ln r + i\theta \; (r > 0, \alpha < \theta < \alpha + 2\pi)$ of the logarithmic function is analytic at each point z in the stated domain, obtain its derivative by differentiating each side of the identity (Sec. 30)

$$e^{\log z} = z \qquad (z \neq 0)$$

and using the chain rule.

7. Find all roots of the equation $\log z = i\pi/2$.

Ans. $z = i$.

8. Suppose that the point $z = x + iy$ lies in the horizontal strip $\alpha < y < \alpha + 2\pi$. Show that when the branch $\log z = \ln r + i\theta \; (r > 0, \alpha < \theta < \alpha + 2\pi)$ of the logarithmic function is used, $\log(e^z) = z$. [Compare with equation (4), Sec. 30.]

9. Show that

(a) the function $f(z) = \text{Log}(z - i)$ is analytic everywhere except on the portion $x \leq 0$ of the line $y = 1$;

(b) the function

$$f(z) = \frac{\text{Log}(z+4)}{z^2 + i}$$

is analytic everywhere except at the points $\pm(1-i)/\sqrt{2}$ and on the portion $x \leq -4$ of the real axis.

10. Show in two ways that the function $\ln(x^2+y^2)$ is harmonic in every domain that does not contain the origin.

11. Show that

$$\operatorname{Re}[\log(z-1)] = \frac{1}{2}\ln[(x-1)^2+y^2] \qquad (z \neq 1).$$

Why must this function satisfy Laplace's equation when $z \neq 1$?

32. SOME IDENTITIES INVOLVING LOGARITHMS

If z_1 and z_2 denote any two nonzero complex numbers, it is straightforward to show that

$$\log(z_1 z_2) = \log z_1 + \log z_2. \tag{1}$$

This statement, involving a multiple-valued function, is to be interpreted in the same way that the statement

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2 \tag{2}$$

was in Sec. 8. That is, if values of two of the three logarithms are specified, then there is a value of the third such that equation (1) holds.

The verification of statement (1) can be based on statement (2) in the following way. Since $|z_1 z_2| = |z_1||z_2|$ and since these moduli are all positive real numbers, we know from experience with logarithms of such numbers in calculus that

$$\ln|z_1 z_2| = \ln|z_1| + \ln|z_2|.$$

So it follows from this and equation (2) that

$$\ln|z_1 z_2| + i\arg(z_1 z_2) = (\ln|z_1| + i\arg z_1) + (\ln|z_2| + i\arg z_2). \tag{3}$$

Finally, because of the way in which equations (1) and (2) are to be interpreted, equation (3) is the same as equation (1).

EXAMPLE. To illustrate statement (1), write $z_1 = z_2 = -1$ and recall from Examples 2 and 3 in Sec. 30 that

$$\log 1 = 2n\pi i \quad \text{and} \quad \log(-1) = (2n+1)\pi i,$$

where $n = 0, \pm 1, \pm 2, \ldots$. Noting that $z_1 z_2 = 1$ and using the values

$$\log(z_1 z_2) = 0 \quad \text{and} \quad \log z_1 = \pi i,$$

EXERCISES

1. Show that if $\operatorname{Re} z_1 > 0$ and $\operatorname{Re} z_2 > 0$, then

$$\operatorname{Log}(z_1 z_2) = \operatorname{Log} z_1 + \operatorname{Log} z_2.$$

Suggestion: Write $\Theta_1 = \operatorname{Arg} z_1$ and $\Theta_2 = \operatorname{Arg} z_2$. Then observe how it follows from the stated restrictions on z_1 and z_2 that $-\pi < \Theta_1 + \Theta_2 < \pi$.

2. Show that for any two nonzero complex numbers z_1 and z_2,

$$\operatorname{Log}(z_1 z_2) = \operatorname{Log} z_1 + \operatorname{Log} z_2 + 2N\pi i$$

where N has one of the values $0, \pm 1$. (Compare with Exercise 1.)

3. Verify expression (4), Sec. 32, for $\log(z_1/z_2)$ by

(a) using the fact that $\arg(z_1/z_2) = \arg z_1 - \arg z_2$ (Sec. 8);

(b) showing that $\log(1/z) = -\log z$ $(z \neq 0)$, in the sense that $\log(1/z)$ and $-\log z$ have the same set of values, and then referring to expression (1), Sec. 32, for $\log(z_1 z_2)$.

4. By choosing specific nonzero values of z_1 and z_2, show that expression (4), Sec. 32, for $\log(z_1/z_2)$ is not always valid when *log* is replaced by *Log*.

5. Show that property (6), Sec. 32, also holds when n is a negative integer. Do this by writing $z^{1/n} = (z^{1/m})^{-1}$ $(m = -n)$, where n has any one of the negative values $n = -1, -2, \ldots$ (see Exercise 9, Sec. 10), and using the fact that the property is already known to be valid for positive integers.

6. Let z denote any nonzero complex number, written $z = re^{i\Theta}$ $(-\pi < \Theta \leq \pi)$, and let n denote any fixed positive integer $(n = 1, 2, \ldots)$. Show that all of the values of $\log(z^{1/n})$ are given by the equation

$$\log(z^{1/n}) = \frac{1}{n} \ln r + i \frac{\Theta + 2(pn + k)\pi}{n},$$

where $p = 0, \pm 1, \pm 2, \ldots$ and $k = 0, 1, 2, \ldots, n - 1$. Then, after writing

$$\frac{1}{n} \log z = \frac{1}{n} \ln r + i \frac{\Theta + 2q\pi}{n},$$

where $q = 0, \pm 1, \pm 2, \ldots$, show that the set of values of $\log(z^{1/n})$ is the same as the set of values of $(1/n) \log z$. Thus show that $\log(z^{1/n}) = (1/n) \log z$ where, corresponding to a value of $\log(z^{1/n})$ taken on the left, the appropriate value of $\log z$ is to be selected on the right, and conversely. [The result in Exercise 5*(a)*, Sec. 31, is a special case of this one.]

Suggestion: Use the fact that the remainder upon dividing an integer by a positive integer n is always an integer between 0 and $n - 1$, inclusive; that is, when a positive integer n is specified, any integer q can be written $q = pn + k$, where p is an integer and k has one of the values $k = 0, 1, 2, \ldots, n - 1$.

In view of the periodicity of $\sin z$ and $\cos z$, it follows immediately from relaions (4) that $\sinh z$ and $\cosh z$ are periodic with period $2\pi i$. Relations (4), together with statements (17) and (18) in Sec. 34, also tell us that

$$\sinh z = 0 \quad \text{if and only if} \quad z = n\pi i \ (n = 0, \pm 1, \pm 2, \ldots) \tag{14}$$

and

$$\cosh z = 0 \quad \text{if and only if} \quad z = \left(\frac{\pi}{2} + n\pi\right) i \ (n = 0, \pm 1, \pm 2, \ldots). \tag{15}$$

The hyperbolic tangent of z is defined by means of the equation

$$\tanh z = \frac{\sinh z}{\cosh z} \tag{16}$$

and is analytic in every domain in which $\cosh z \neq 0$. The functions $\coth z$, $\operatorname{sech} z$, and $\operatorname{csch} z$ are the reciprocals of $\tanh z$, $\cosh z$, and $\sinh z$, respectively. It is straightforward to verify the following differentiation formulas, which are the same as those established in calculus for the corresponding functions of a real variable:

$$\frac{d}{dz} \tanh z = \operatorname{sech}^2 z, \qquad \frac{d}{dz} \coth z = -\operatorname{csch}^2 z, \tag{17}$$

$$\frac{d}{dz} \operatorname{sech} z = -\operatorname{sech} z \tanh z, \qquad \frac{d}{dz} \operatorname{csch} z = -\operatorname{csch} z \coth z. \tag{18}$$

EXERCISES

1. Verify that the derivatives of $\sinh z$ and $\cosh z$ are as stated in equations (2), Sec. 35.

2. Prove that $\sinh 2z = 2 \sinh z \cosh z$ by starting with

(a) definitions (1), Sec. 35, of $\sinh z$ and $\cosh z$;

(b) the identity $\sin 2z = 2 \sin z \cos z$ (Sec. 34) and using relations (3) in Sec. 35.

3. Show how identities (6) and (8) in Sec. 35 follow from identities (9) and (6), respectively, in Sec. 34.

4. Write $\sinh z = \sinh(x + iy)$ and $\cosh z = \cosh(x + iy)$, and then show how expressions (9) and (10) in Sec. 35 follow from identities (7) and (8), respectively, in that section.

5. Verify expression (12), Sec. 35, for $|\cosh z|^2$.

6. Show that $|\sinh x| \leq |\cosh z| \leq \cosh x$ by using

(a) identity (12), Sec. 35;

(b) the inequalities $|\sinh y| \leq |\cos z| \leq \cosh y$, obtained in Exercise 9*(b)*, Sec. 34.

7. Show that

(a) $\sinh(z + \pi i) = -\sinh z$; *(b)* $\cosh(z + \pi i) = \cosh z$;
(c) $\tanh(z + \pi i) = \tanh z$.

become single-valued and analytic because they are then compositions of analytic functions.

The derivatives of these three functions are readily obtained from their logarithmic expressions. The derivatives of the first two depend on the values chosen for the square roots:

$$\frac{d}{dz}\sin^{-1} z = \frac{1}{(1-z^2)^{1/2}}, \tag{5}$$

$$\frac{d}{dz}\cos^{-1} z = \frac{-1}{(1-z^2)^{1/2}}. \tag{6}$$

The derivative of the last one,

$$\frac{d}{dz}\tan^{-1} z = \frac{1}{1+z^2}, \tag{7}$$

does not, however, depend on the manner in which the function is made single-valued.

Inverse hyperbolic functions can be treated in a corresponding manner. It turns out that

$$\sinh^{-1} z = \log\left[z + (z^2+1)^{1/2}\right], \tag{8}$$

$$\cosh^{-1} z = \log\left[z + (z^2-1)^{1/2}\right], \tag{9}$$

and

$$\tanh^{-1} z = \frac{1}{2}\log\frac{1+z}{1-z}. \tag{10}$$

Finally, we remark that common alternative notation for all of these inverse functions is arcsin z, etc.

EXERCISES

1. Find all the values of

(a) $\tan^{-1}(2i)$; *(b)* $\tan^{-1}(1+i)$; *(c)* $\cosh^{-1}(-1)$; *(d)* $\tanh^{-1} 0$.

Ans. *(a)* $\left(n+\frac{1}{2}\right)\pi + \frac{i}{2}\ln 3$ $(n = 0, \pm 1, \pm 2, \ldots)$;

(d) $n\pi i$ $(n = 0, \pm 1, \pm 2, \ldots)$.

2. Solve the equation $\sin z = 2$ for z by

(a) equating real parts and then imaginary parts in that equation;

(b) using expression (2), Sec. 36, for $\sin^{-1} z$.

Ans. $z = \left(2n+\frac{1}{2}\right)\pi \pm i\ln(2+\sqrt{3})$ $(n = 0, \pm 1, \pm 2, \ldots)$.

3. Solve the equation $\cos z = \sqrt{2}$ for z.

4. Derive formula (5), Sec. 36, for the derivative of $\sin^{-1} z$.

5. Derive expression (4), Sec. 36, for $\tan^{-1} z$.

6. Derive formula (7), Sec. 36, for the derivative of $\tan^{-1} z$.

7. Derive expression (9), Sec. 36, for $\cosh^{-1} z$.