9.6 Exercises

1. Express the following complex numbers in the form $a + bi$.
   (a) $(1 + i)^2$.
   (b) $1/i$.
   (c) $1/(1 + i)$.
   (d) $(2 + 3i)(3 - 4i)$.
   (e) $(1 + i)/(1 - 2i)$.
   (f) $i^5 + i^{16}$.
   (g) $1 + i + i^2 + i^3$.
   (h) $\frac{1}{2}(1 + i)(1 + i^{-8})$.
2. Compute the absolute values of the following complex numbers.
   (a) $1 + i$.
   (b) $3 + 4i$.
   (c) $(1 + i)/(1 - i)$.
   (d) $1 + i + i^2$.
   (e) $i^7 + i^{10}$.
   (f) $2(1 - i) + 3(2 + i)$.
3. Compute the modulus and principal argument of each of the following complex numbers.
   (a) $2i$.
   (b) $-3i$.
   (c) -1.
   (d) 1.
   (e) $-3 + \sqrt{3}\, i$.
   (f) $(1 + i)/\sqrt{2}$.
   (g) $(-1 + i)^3$.
   (h) $(-1 - i)^3$.
   (i) $1/(1 + i)$.
   (j) $1/(1 + i)^2$.
4. In each case, determine all real numbers x and y which satisfy the given relation.
   (a) $x + iy = x - iy$.
   (b) $x + iy = |x + iy|$.
   (c) $|x + iy| = |x - iy|$.
   (d) $(x + iy)^2 = (x - iy)^2$.
   (e) $\dfrac{x + iy}{x - iy} = x - iy$.
   (f) $\displaystyle\sum_{k=0}^{100} i^k = x + iy$.
5. Make a sketch showing the set of all z in the complex plane which satisfy each of the following conditions.
   (a) $|z| < 1$.
   (b) $z + \bar{z} = 1$.
   (c) $z - \bar{z} = i$.
   (d) $|z - 1| = |z + 1|$.
   (e) $|z - i| = |z + i|$.
   (f) $z + \bar{z} = |z|^2$.
6. Let f be a polynomial with real coefficients.
   (a) Show that $\overline{f(z)} = f(\bar{z})$ for every complex z.
   (b) Use part (a) to deduce that the nonreal zeros of f (if any exist) must occur in pairs of conjugate complex numbers.
7. Prove that an ordering relation cannot be introduced in the complex number system so that all three order axioms of Section I3.4 are satisfied.

   [*Hint:* Assume that such an ordering can be introduced and try to decide whether the imaginary unit i is positive or negative.]

8. Define the following "pseudo-ordering" among the complex numbers. If $z = x + iy$, we say that z is positive if and only if $x > 0$. Which of the order axioms of Section I3.4 are satisfied with this definition of positive?
9. Solve Exercise 8 if the pseudo-ordering is defined as follows: We say that z is positive if and only if $|z| > 0$.
10. Solve Exercise 8 if the pseudo-ordering is defined as follows: If $z = x + iy$, we say that z is positive if and only if $x > y$.
11. Make a sketch showing the set of all complex z which satisfy each of the following conditions.
    (a) $|2z + 3| < 1$.
    (b) $|z + 1| < |z - 1|$.
    (c) $|z - i| \le |z + i|$.
    (d) $|z| \le |2z + 1|$.
12. Let $w = (az + b)/(cz + d)$, where a, b, c, and d are real. Prove that

$$w - \bar{w} = (ad - bc)(z - \bar{z})/|cz + d|^2 .$$

    If $ad - bc > 0$, prove that the imaginary parts of z and w have the same sign.

$(-\infty, +\infty)$ *by the equation* $f(x) = e^{tx}$ *are solutions of the differential equation* (9.16) *if and only if* t *is a root of the characteristic equation*

$$t^2 + at + b = 0 .$$

Proof. Let $L(y) = y'' + ay' + by$. Since $f'(x) = te^{tx}$, we also have $f''(x) = t^2e^{tx}$, so $L(f) = e^{tx}(t^2 + at + b)$. But e^{tx} is never zero since $e^{tx}e^{-tx} = e^0 = 1$. Hence, $L(f) = 0$ if and only if $t^2 + at + b = 0$. But if we write $f = u + iv$, we find $L(f) = L(u) + iL(v)$, and hence $L(f) = 0$ if and only if both $L(u) = 0$ and $L(v) = 0$. This completes the proof.

Note: If $t = \alpha + i\beta$, the real and imaginary parts of f are given by (9.14). If the characteristic equation has two distinct roots, real or complex, the linear combination

$$y = c_1u(x) + c_2v(x)$$

is the general solution of the differential equation. This agrees with the results proved in Theorem 8.7.

Further examples of complex functions are discussed in the next set of exercises.

9.10 Exercises

1. Express each of the following complex numbers in the form $a + bi$.
   (a) $e^{\pi i/2}$.
   (b) $2e^{-\pi i/2}$.
   (c) $3e^{\pi i}$.
   (d) $-e^{-\pi i}$.
   (e) $i + e^{2\pi i}$.
   (f) $e^{\pi i/4}$.
   (g) $e^{\pi i/4} - e^{-\pi i/4}$.
   (h) $\dfrac{1 - e^{\pi i/2}}{1 + e^{\pi i/2}}$.
2. In each case, find all real x and y that satisfy the given relation.
   (a) $x + iy = xe^{iy}$.
   (b) $x + iy = ye^{ix}$.
   (c) $e^{x+iy} = -1$.
   (d) $\dfrac{1 + i}{1 - i} = xe^{iy}$.
3. (a) Prove that $e^z \neq 0$ for all complex z.
   (b) Find all complex z for which $e^z = 1$.
4. (a) If θ is real, show that

$$\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad \text{and} \qquad \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} .$$

   (b) Use the formulas in (a) to deduce the identities

$$\cos^2 \theta = \tfrac{1}{2}(1 + \cos 2\theta), \qquad \sin^2 \theta = \tfrac{1}{2}(1 - \cos 2\theta) .$$

5. (a) Prove *DeMoivre's theorem*,

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta ,$$

   valid for every real θ and every positive integer n.
   (b) Take $n = 3$ in part (a) and deduce the trigonometric identities

$$\sin 3\theta = 3 \cos^2 \theta \sin \theta - \sin^3 \theta, \qquad \cos 3\theta = \cos^3 \theta - 3 \cos \theta \sin^2 \theta .$$

6. Prove that every trigonometric sum of the form

$$S_n(x) = \tfrac{1}{2}a_0 + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx)$$

can be expressed as a sum of complex exponentials,

$$S_n(x) = \sum_{k=-n}^{n} c_k e^{ikx},$$

where $c_k = \frac{1}{2}(a_k - ib_k)$ for $k = 1, 2, \ldots, n$. Determine corresponding formulas for c_{-k}.

7. (a) If m and n are integers, prove that

$$\int_0^{2\pi} e^{inx} e^{-imx}\, dx = \begin{cases} 0 & \text{if } m \neq n, \\ 2\pi & \text{if } m = n. \end{cases}$$

(b) Use part (a) to deduce the orthogonality relations for the sine and cosine (m and n are integers, $m^2 \neq n^2$):

$$\int_0^{2\pi} \sin nx \cos mx\, dx = \int_0^{2\pi} \sin nx \sin mx\, dx = \int_0^{2\pi} \cos nx \cos mx\, dx = 0,$$

$$\int_0^{2\pi} \sin^2 nx\, dx = \int_0^{2\pi} \cos^2 nx\, dx = \pi \qquad \text{if } n \neq 0.$$

8. Given a complex number $z \neq 0$. Write $z = re^{i\theta}$, where $\theta = \arg(z)$. Let $z_1 = Re^{i\alpha}$, where $R = r^{1/n}$ and $\alpha = \theta/n$, and let $\epsilon = e^{2\pi i/n}$, where n is a positive integer.
(a) Show that $z_1^n = z$; that is, z_1 is an nth root of z.
(b) Show that z has exactly n distinct nth roots,

$$z_1, \epsilon z_1, \epsilon^2 z_1, \ldots, \epsilon^{n-1} z_1,$$

and that they are equally spaced on a circle of radius R.
(c) Determine the three cube roots of i.
(d) Determine the four fourth roots of i.
(e) Determine the four fourth roots of $-i$.

9. The definitions of the sine and cosine functions can be extended to the complex plane as follows:

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

When z is real, these formulas agree with the ordinary sine and cosine functions. (See Exercise 4.) Use these formulas to deduce the following properties of complex sines and cosines. Here u, v, and z denote complex numbers, with $z = x + iy$.
(a) $\sin(u + v) = \sin u \cos v + \cos u \sin v$.
(b) $\cos(u + v) = \cos u \cos v - \sin u \sin v$.
(c) $\sin^2 z + \cos^2 z = 1$.
(d) $\cos(iy) = \cosh y, \qquad \sin(iy) = i \sinh y$.
(e) $\cos z = \cos x \cosh y - i \sin x \sinh y$.
(f) $\sin z = \sin x \cosh y + i \cos x \sinh y$.