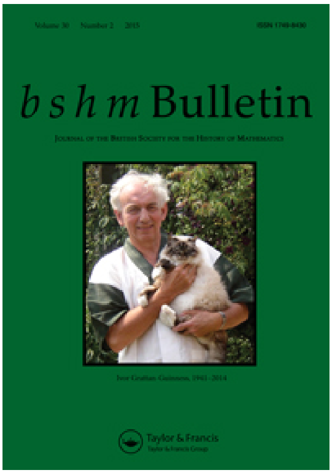

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Vito Volterra's construction of a nonconstant function with a bounded, non-Riemann integrable derivative

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In the 1880s the research on the theory of integration was focused mainly on the properties of infinite sets. The development of nowhere dense sets with positive outer content, known nowadays as *nowhere dense sets with positive measure*, allowed the construction of general functions with the purpose of extending Riemann's definition of integral. Vito Volterra provided, in 1881, an example of a differentiable function F whose derivative F' is bounded but not Riemann integrable. In this article we present and discuss Volterra's example.

Introduction

Vito Volterra (1860–1940) was an Italian mathematician generally known for his contributions in different fields such as mathematical analysis, mathematical physics, celestial mechanics, the mathematical theory of elasticity and mathematical biometrics. According to the *Complete dictionary of scientific biography* (see Volterra 2008, 85), some of his major works in these fields included the foundation of the theory of functionals and the solution of the type of integral equations with variable limits that now bear his name, methods of integrating hyperbolic partial differential equations, the motion of the Earth's poles and the list goes on.

In this article we will turn our attention to one particular aspect of Volterra's work on mathematical analysis. In 1881, he published the construction of nowhere dense sets of positive outer content. In the same year, he also published an example of a function with a bounded, non-Riemann integrable derivative. This example is remarkable since it shows that within the context of Riemann's theory of integration, the fundamental operations of differentiation and integration are not entirely reversible. Now, before introducing Volterra's example, we would like to present a brief historical background about the integral calculus and its evolution.

Historical background: the integral calculus in the eighteenth century

In the eighteenth century, the *integral calculus* was considered essentially as the inverse process of *differential calculus*. In other words, integration was conceived as the reverse process of differentiation (also called *antidifferentiation*). This is largely due to the influence of Newton's and Leibniz's works. In the first instance, Newton solved area and volume problems interpreting them as inverse problems of rate of change. By contrast, Leibniz tackled the same problems interpreting them as a sum of differentials. Of course, both ultimately can be considered as antidifferentiation (see Edwards 1979, 266).

Evidence of the above mentioned can be found in textbooks of calculus written in Europe in the eighteenth and nineteenth century. Leonhard Euler (1707–83), for example, begins his treatise about integral calculus with the following:

> **Definition**: Integral calculus is the method of finding, from a given differential, the quantity itself, and the operation that produces this is generally called integration. (Euler 1992, 5, Vol. 11)

Another example is the book *Instituzioni analitiche use ad della Italian gioventu* from 1748, written by the Italian mathematician Maria Gaetana Agnesi (1718–99). This treatise is considered as one of the first textbooks of Calculus (Truesdell 1989). Agnesi treated jointly differential and integral calculus, emphasizing its nature as inverse problems. In book three (volume II) we find the following:

> The Integral Calculus, which is also used to be called *Summatory Calculus*, is the method of reducing a differential or fluxion quantity, to that quantity of which it is the difference or fluxion. Whence the operations of Integral Calculus are just the contrary to those of the Differential; and therefore it is also called The Inverse Method of Fluxions or Differences. (Agnesi 1748, 613)

It wasn't until the 1820s, with the introduction of the concept of limit, that the integral calculus was considered from another standpoint. The French mathematician Augustin Louis Cauchy (1789–1857) introduced a new perspective to consider the processes of integration and differentiation within an analytical context, since he considered that the concepts of *integral* and *derivative* should be defined independently from each other.

In his *Résumé des leçons donnés à l'École Royale Polytechnique sur le calcul infinitésimale* from 1823 (see Cauchy 1899), Cauchy defined the derivative of $f(x)$ as the limit of the differential quotient

$$\frac{f(x+h)-f(x)}{h}$$

when h is infinitely small (or tends to zero). On the other hand, he introduced the definition of definite integral, emphasizing the need to establish its existence regardless of antidifferentiation. Suppose that $f(x)$ is a continuous function defined on an interval $[x_0, X]$. Consider the partition of this interval $x_0, x_1, x_2, \ldots, x_{n-1}, x_n = X$, and the sum

$$S_n = \sum_{i=1}^{n} f(x_{i-1})(x_i - x_{i-1}).$$

(also called the *Sum of Cauchy*). If the absolute values of the differences $x_i - x_{i-1}$ decrease indefinitely, the value of S_n tends to a certain limit S. Cauchy called this limit the definite integral and he represented it with the symbol

$$\int_{x_0}^{X} f(x)\mathrm{d}x.$$

Once the concepts of integral and derivative were established independently, Cauchy was able to bind them through a series of results that eventually lead to what we know today as the fundamental theorem of integral calculus for continuous functions. Cauchy's definition of integral was a significant development for two important reasons: First, the integral is defined as a *limit*, and second, its *existence* does not depend on antidifferentiation.

Cauchy not only proved the existence of the integral for any continuous function but he also treated cases when the integrand $f(x)$ is bounded on $[a, b]$ and is discontinuous at the point $x_0 \in [a, b]$; Cauchy defined

$$\int_a^b f(x)\mathrm{d}x = \lim_{\varepsilon \to 0^+} \int_a^{x_0 - \varepsilon} f(x)\mathrm{d}x + \lim_{\varepsilon \to 0^+} \int_{x_0 + \varepsilon}^b f(x)\mathrm{d}x$$

when these limits exist. The definite integral for a function with any finite number of discontinuities in $[a, b]$ can be defined analogously.

The problem of integrating discontinuous functions did not become apparent until the modern concept of function was recognized. The German mathematicians Peter Gustav Lejeune Dirichlet (1805–59) and Georg Friedrich Bernhard Riemann (1826–66) made important contributions to solve this problem. On the one hand, Dirichlet was the first to call attention to the existence of functions that are discontinuous on an infinite set of points in a finite interval and to the problem of extending the concept of the integral to these kinds of functions. On the other hand, Riemann tackled the issue of integrability for more general functions in his work of 1854: *Ueber die einer Function durch eine Darstellbarkeit trigonometrische Reihe* (Riemann 1902).

In brief, Riemann developed a theory of integration, based on the ideas of Cauchy, and reduced as much as possible the assumptions on the functions to be integrated. Riemann also established necessary and sufficient conditions for the integrability of bounded functions over an interval $[a, b]$. He considered a partition $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ on the interval $[a, b]$ and defined the oscillation of $f(x)$ in $[x_{i-1}, x_i]$ as the difference between the greatest and least value of $f(x)$ on $[x_{i-1}, x_i]$. Then he proved that a necessary and sufficient condition is that the sums

$$S_n = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}),$$

where $t_i \in [x_{i-1}, x_i]$, approach a unique limit (that the integral exists) as the maximum $|x_i - x_{i-1}|$ approaches zero, and that the sum of the intervals $[x_{i-1}, x_i]$ in which the oscillation of $f(x)$ is greater than any given number δ must approach zero with the size of the intervals. Riemann denoted this limit by

$$\int_a^b f(x)\mathrm{d}x.$$

Remark 1: Nowadays, the oscillation of a function is defined in terms of the *supremum* and *infimum*. If f is bounded on $[a, b]$, the *oscillation of f* on $[a, b]$ is defined by

$$W_f[a, b] = \sup\{f(x) \mid a \le x \le b\} - \inf\{f(x) \mid a \le x \le b\},$$

which can also be written as

$$W_f[a,b] = \sup\{\ |f(x) - f(x')| \mid a \le x,\ x' \le b\}.$$

If $a < x < b$, the *oscillation of f at x* is defined by

$$w_f(x) = \lim_{h\to 0^+} W_f(x-h, x+h).$$

The corresponding definitions for $x = a$ and $x = b$ are

$$w_f(a) = \lim_{h\to 0^+} W_f(a, a+h) \qquad \text{and} \qquad w_f(b) = \lim_{h\to 0^+} W_f(b-h, b).$$

For a fixed x in (a,b), $W_f(x-h, x+h)$ is a nonnegative and nondecreasing function of h for $0 < h < \min(x-a, b-x)$. This means that the function $w_f(x)$ exists and is nonnegative. Similar arguments apply for $w_f(a)$ and $w_f(b)$.

Remark 2: Let f be defined on $[a,b]$. If D denotes the set of discontinuities of f then

$$D = \{x \in [a,b] \mid w_f(x) > 0\}.$$

Historically, Riemann's definition of integral represented a significant advance because it was established in terms of more general summations (called now *Riemann sums*) and it involved the modern conception of function. Therefore it applies not only to continuous functions, but also to functions possessing an infinity of points of discontinuity in every interval. As an example of this, Riemann himself constructed the next function:

$$f(x) = \sum_{i=1}^{\infty} \frac{(nx)}{n^2},$$

where $(x) = x - m(x)$, if $x \neq 2k + 1/2$, and $m(x)$ is the integer nearest x with $(x) = 0$ if $x = 2k + 1/2$. This function has an infinite number of discontinuities in an even arbitrarily small interval yet it meets Riemann integrability conditions.

During the decade of 1880 research on the theory of integration was focused mainly on the properties of infinite sets. The development of nowhere dense sets with positive outer content, allowed the construction of more general functions with the purpose of extending Riemann's definition of integral.

Remark 3: A subset C of a topological space X is nowhere dense if and only if each neighbourhood of each point $x \in X$ contains a point y such that the complement of C contains y and a neighbourhood of y (see Viro *et al.* 2008).

Remark 4: If S denotes a bounded set of real numbers, the *outer content* of S is the real number $c_e(S)$ defined as follows. Let $I_1, I_2, \ldots, I_n$ denote a finite set of intervals which cover S in the sense that $S \subset \cup_{k=1}^{n} I_k$. Then $c_e(S)$ denotes the greatest lower bound of all real numbers of the form $\sum_{k=1}^{n} L(I_k)$, where $L(I_k)$ is the length of I_k. Similarly, the *inner content* of S, $c_i(S)$, is defined as the least upper bound of $\sum_{k=1}^{n} L(I_k)$, where $I_k \cap I_k' = \emptyset$ and $\cup_{k=1}^{n} I_k \subset S$.

Among the mathematicians who contributed to the theory of integration and the study of properties of infinite sets was Vito Volterra. In a paper written in 1881 while he was still a student at the *Scuola Normale Superiore* de Pisa, Volterra published the construction of a closed nowhere dense set in $[0,1]$ with outer content greater than $2/3$ (Volterra 1881a). Using the characteristic function defined on this set, Volterra exhibited a non-integrable, pointwise discontinuous function.

Remark 5: A function f is *pointwise discontinuous* if it has infinitely many points of discontinuity yet is continuous on a dense set.

In the same year, Volterra also published an example of a continuous, non-constant function with bounded, nonintegrable derivative (Volterra 1881b). This example became part of a set of functions, usually called *pathological*, that would defy the intuition of mathematicians while setting the stage for further developments in the theory of integration.

Construction of a continuous, nonconstant function with bounded, nonintegrable derivative

This section describes the concepts in Volterra's example. First, we give the iterative construction of the nowhere dense set with positive outer content. Then we define a function whose derivative is bounded but it is not Riemann integrable.

A nowhere dense set with positive outer content

Let us subdivide the interval $[0,1]$ into an infinite number of intervals. The first step is to construct a sequence $1 > c_1 > c_2 > \cdots$ with the following properties:

1. The sequence $c_n \to 0$, when $n \to \infty$
2. $1 - c_1 = \frac{1}{2^{2\cdot 1}}(1-0)$
3. $c_n - c_{n+1} < 1 - c_1$

In the next step we exclude the interval $(c_1, 1)$ and we apply the same procedure to each of the intervals (c_{n+1}, c_n). That is, we construct a sequence $1 > c^1_{n,1} > c^1_{n,2} > c^1_{n,3} > \cdots$ satisfying the following:

1. $c^1_{n,k} \to c_{n+1}$, when $k \to \infty$
2. $c_n - c^1_{n,1} = \frac{1}{2^{2\cdot 2}}(c_n - c_{n+1})$
3. $c^1_{n,k} - c^1_{n,k+1} < c_n - c^1_{n,1}$

According to this, each one of the intervals (c_{n+1}, c_n) is divided into an infinite number of intervals $(c^1_{n,k+1}, c^1_{n,k})$, for $k = 1, 2, 3, \ldots$. Then the above procedure is applied for each of these intervals, excluding $(c^1_{n,1}, c_n)$, and so on *ad infinitum*. Figure 1 shows a diagram representing the first three steps of the subdivision of $[0,1]$.

Let us denote by $\overline{C}$ the set of points of division c's together with all their limit points including the points 0 and 1. Volterra proved that $\overline{C}$ possess the following property: *for* $s \in [0,1]\backslash\overline{C}$ *there exists an open interval* (a,b) *such that* $x \in (a,b)$, *with* $a, b \in \overline{C}$. In Volterra's words 'if we consider any portion of the interval $[0,1]$ in that

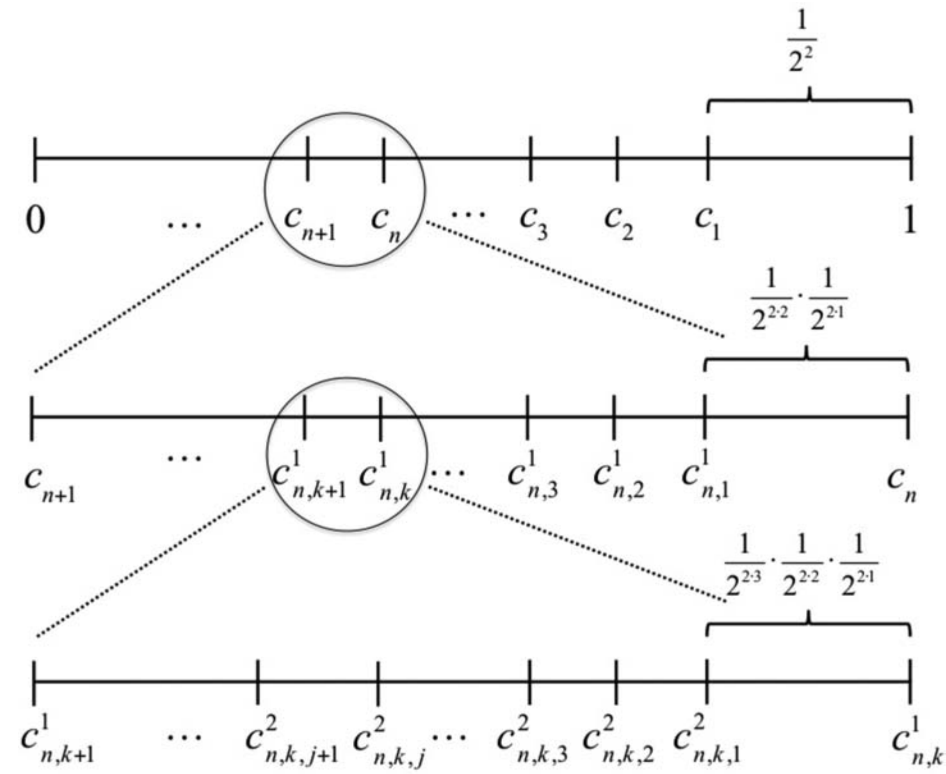

Figure 1. The first three steps of the subdivision of the interval [0, 1]

portion it is possible to find an interval on which there are not points of $\overline{C}$' (Volterra 1881a, 81–82). This is equivalent to saying that $\overline{C}$ is a nowhere dense set.

On the other hand, notice that the set $X = [0,1]\backslash\overline{C}$ is a disjoint union of open intervals whose end points belongs to $\overline{C}$. In fact, these open intervals are precisely those intervals which are excluded in the subdivision process from the construction of $\overline{C}$. That is, the intervals

$$(c_1, 1), \left(c^1_{1,1}, c_1\right), \left(c^1_{2,1}, c_2\right), \ldots, \left(c^1_{n,1}, c_n\right), \ldots$$

Moreover, if s_n denotes the total length of those intervals forming X that have length greater or equal to $1/2^{2n}$, then we have

$$s_n < \frac{1}{2^2} + \frac{1}{2^4} + \cdots + \frac{1}{2^{2n}} < \frac{1}{3},$$

which follows from the construction of the set $\overline{C}$. From the above relationship we can deduce that the sum of those intervals containing elements of the set $\overline{C}$ is greater than $2/3$, that is, the outer content of $\overline{C}$ is greater than $2/3$. Once the set $\overline{C}$ is constructed, Volterra exhibited a function whose bounded derivative is not integrable.

A function with bounded, nonintegrable derivative

The fundamental idea of Volterra was to associate to each interval $[a, b]$ a function with similar properties as the function defined by

$$f(x) = x^2 \sin\left(\frac{1}{x}\right).$$

For doing this, let $f_{a,b} : [a, b] \to \mathbb{R}$ be defined as follows:

1. $f_{a,b}(x) = 0$, for $x = a$ or $x = b$
2. $f_{a,b}(x) = (x - a)^2 \sin\left(\frac{1}{x-a}\right)$, for $a < x \leq x_1$, where x_1 is the largest number less than or equal to $\frac{a+b}{2}$ for which $(x - a)^2 \sin\left(\frac{1}{x-a}\right)$ has maximum value.
3. $f_{a,b}(x) = (x_1 - a)^2 \sin\left(\frac{1}{x_1-a}\right) = (b - x_2)^2 \sin\left(\frac{1}{b-x_2}\right)$, if $x_1 \leq x \leq x_2$.
4. $f_{a,b}(x) = (b - x)^2 \sin\left(\frac{1}{b-x}\right)$, if $x_2 \leq x < b$, where x_2 is the smallest number greater than or equal to $\frac{a+b}{2}$ for which $(b - x)^2 \sin\left(\frac{1}{b-x}\right)$ has maximum value.

Figure 2 shows a representation of $f_{a,b}$. Notice also that $f_{a,b}(x)$ is bounded. In fact

$$|f_{a,b}(x)| < \begin{cases} (m - a)^2 \\ (b - m)^2 \end{cases} \qquad \forall\, m \in [a, b]. \tag{1}$$

The derivative of $f_{a,b}(x)$ in a neighbourhood of a or b behaves in the same way as the derivative of the function $f(x) = x^2 \sin(1/x)$ in a neighbourhood of 0. In other words, $f'_{a,b}(x)$ oscillates indefinitely between the values -1 and 1, when x approaches the end-points a or b (see Figure 3). This means that $f'_{a,b}(x)$ is discontinuous at a and b. Notice also that

$$|f'_{a,b}(x)| < 2(b - a) + 1.$$

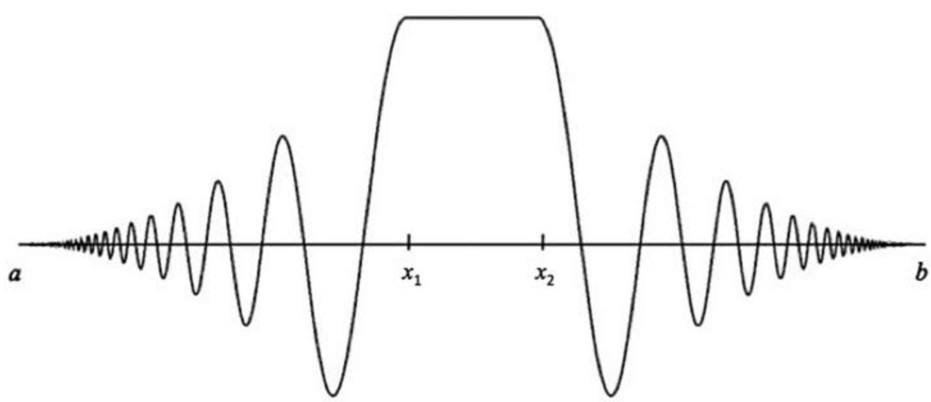

Figure 2. A representation of $f_{a,b}$ on $[a, b]$

Figure 3. Representation of the derivative $f_{a,b}^{'}(x)$

Finally, based on the above, Volterra defined the function $F : [0,1] \to \mathbb{R}$ as follows (Figure 4):

$$F(x) = \begin{cases} f_{a,b}(x), & \text{if } x \in (a,b) \text{ for some interval } (a,b) \subset [0,1]\backslash\overline{C}, \\ 0, & \text{if } x \in \overline{C}. \end{cases}$$

The function $F(x)$ is continuous and differentiable at every $x \in [0,1]$. It is not difficult to prove that $F'(x)$ exists for all $x \in \overline{C}$. Furthermore, $F'(x) = 0$ for $x \in \overline{C}$. To prove this, let $\varepsilon > 0$ and suppose that $|x - t| < \varepsilon$. If $t \in \overline{C}$, then

$$\frac{f(t) - f(x)}{t - x} = 0.$$

If $t \notin \overline{C}$, then t belongs to some open interval (a,b) content in $[0,1]\backslash\overline{C}$. Suppose that a is the endpoint nearest to x. Then, considering (1) from the previous page, we have

$$\left|\frac{f(t) - f(x)}{t - x}\right| = \left|\frac{f(t)}{t - x}\right| \leq \left|\frac{f(t)}{t - a}\right| \leq \frac{|t - a|^2}{|t - a|} < \varepsilon.$$

Hence, $F'(x) = 0$ for $x \in \overline{C}$. Notice also that F' is bounded. In fact, $|F'(x)| \leq 3$, for all $x \in [0,1]$. Figure 5 shows a representation of F'.

At every point of $\overline{C}$, the derivative F' is discontinuous and it has an oscillation equal to 2. Therefore, F' does not fit the Riemann condition to be integrable. In Volterra's words 'in some cases, it can happen that the ordinary definition of the integral is not included in that of Riemann' (Volterra 1881b, 334).

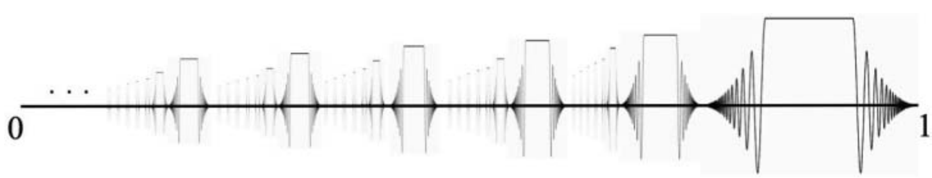

Figure 4. A representation of $F(x)$

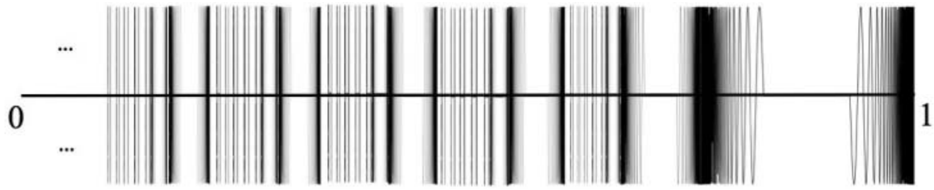

Figure 5. A representation of F'

Nowadays we can use the next criterion for proving that F' is not Riemann integrable:

Theorem 1 (Lebesgue Criterion for Integrability)**:** *A function $f:[a,b]\to\mathbb{R}$ is Riemann integrable if and only if f is bounded and the set of discontinuities of f has Lebesgue measure zero.*

Remark 6: *Measure zero sets* is a term used for naming those sets which have no effect on certain calculations. A set M has *measure zero* if for each $\varepsilon > 0$ there exists a countable collection $\{A_1, A_2, ...\}$ of open intervals such that

1. $M \subset \bigcup_i A_i$,
2. $\sum_i l(A_i) < \varepsilon$,

where $l(A)$ is the length of the interval A.

Final comments

In the mid-nineteenth century mathematical analysis was at the peak of its formative stages. This period was characterized not only for the development of ideas such as continuity, differentiability and integrability but also for the rise of examples of pathological functions that challenged the intuition of several mathematicians.

Volterra's example is one of those pathological functions and we claim that it is the first example of a continuous function possessing a bounded derivative which is not Riemann integrable. This example was surprising since it shows that within the context of Riemann's theory of integration the fundamental operations of differentiation and integration are not entirely reversible. Thus, the process of differentiation could produce bounded functions F' which fails to be integrable in Riemann's sense. In this case, the formula:

$$\int_a^b F'(x)\mathrm{d}x = F(b) - F(a)$$

provided by the fundamental theorem of integral calculus, turns out to be meaningless.

Volterra's example challenged Rieman's approach to integration but also helped to develop further ideas on mathematical analysis. The Lebesgue integral appeared in the next few decades, which could handle many more functions, even though, the Lebesgue integral still could not handle every derivative. It was not until the first half of the twentieth century that we see other definitions of the integral, such as the gauge integral or generalized Riemann integral, that could do the job.

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