# A Zorn's Lemma Proof of the Dimension Theorem for Vector Spaces

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**Abstract.** This note gives a "Zorn's Lemma" style proof that any two bases in a vector space have the same cardinality.

One of the most fundamental notions in linear algebra is that of a basis: A subset $B$ of a vector space $V$ is a *basis* if every element of $V$ is a unique linear combination of elements of $B$. The following theorem of Georg Hamel expresses one of the most important aspects of this definition.

**Dimension Theorem.** *If $V$ is a vector space and $A$ and $B$ each form a basis for $V$, then $|A| = |B|$.*

This common cardinality of a basis for $V$ is of course known as the *dimension* of $V$. Typically, this result is proved in an undergraduate course, but only under the assumption that $A$ is finite. The proof of the general result is usually considered "beyond the scope of the text." One reason for this is presumably that the best-known proofs use some nontrivial facts about cardinal arithmetic for infinite sets (e.g., any infinite set is equinumerous with the set of all of its finite subsets). While important in their own right, these facts are nevertheless tangential from the point of view of linear algebra. The purpose of this note is to give a self-contained Zorn's Lemma-style proof of the Dimension Theorem, in the same spirit that it is so often used in more advanced undergraduate courses. We will take the finite instance of the Dimension Theorem for granted and treat it as a black box.

Let's begin with the definition of the partial ordering to which we will apply Zorn's lemma; once this simple definition is in place, the rest of the proof follows naturally. Define $\mathbb{P}$ to be the set consisting of all linear transformations $T$ from a subspace $W$ of $V$ onto $W$ such that:

1. $A \cap W$ and $B \cap W$ are each a basis for $W$, and
2. the restriction of $T$ to $A \cap W$ is a bijection between $A \cap W$ and $B \cap W$.

This set is equipped with a natural partial ordering: Define $S \leq T$ to mean that the domain of $S$ is included in the domain of $T$ and the restriction of $T$ to the domain of $S$ equals $S$.

Notice that $\mathbb{P}$ is nonempty, since the transformation $0 \mapsto 0$ defined on the trivial subspace $\{0\}$ of $V$ is in $\mathbb{P}$. It is not immediately obvious, however, that $\mathbb{P}$ contains any other elements even if $V$ is nontrivial; we will remark more on this at the end of the proof.

Recall that Zorn's Lemma is the following assertion.

**Zorn's Lemma.** *If $\mathbb{P}$ is a partially-ordered set in which every totally ordered subset of $\mathbb{P}$ has an upper bound in $\mathbb{P}$, then $\mathbb{P}$ has a maximal element.*

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The proof that the partial order $\mathbb{P}$ we defined above isn't trivial is closely related to our main task: demonstrating that if $T : W \to W$ is a maximal element of $\mathbb{P}$, then $W$ must equal $V$.

First we will prove, however, that $\mathbb{P}$ has a maximal element. This is easy and completely standard. If $\mathcal{C} \subseteq \mathbb{P}$ is totally ordered, let $T$ be the function whose domain $W$ is the union of the domains of elements of $\mathcal{C}$ and that satisfies $Tu = v$ if there is an element $S$ of $\mathcal{C}$ such that $Su = v$. It is easy to verify that, since $\mathcal{C}$ is totally ordered, $W$ is a subspace of $V$ spanned by both $A \cap W$ and $B \cap W$, $T$ is well defined (i.e., the definition of $Tu$ does not depend on which element $S$ of $\mathcal{C}$ is used in its definition), and that $T$ defines a linear transformation from $W$ to $W$, which maps $A \cap W$ bijectively to $B \cap W$.

Thus, we may apply Zorn's Lemma and find a $T : W \to W$, which is a maximal element of $\mathbb{P}$. Our task now will be to prove that $W = V$. Suppose that this is not the case and construct increasing sequences $A_1, A_2, \ldots$ and $B_1, B_2, \ldots$ of finite subsets of $A$ and $B$ respectively, such that for all natural numbers $n$:

- $A_1$ is not contained in $W$,
- either $A_n \cup W$ spans $V$ or else $A_n$ is properly contained in $A_{n+1}$,
- $A_n$ is contained in the span of $B_n$, and
- $B_n$ is contained in the span of $A_{n+1}$.

Define

$$A_\infty = (A_1 \cup A_2 \cup \ldots) \setminus W$$

and

$$B_\infty = (B_1 \cup B_2 \cup \ldots) \setminus W.$$

There are two cases to consider, depending on whether $A_\infty$ is finite or infinite. If $A_\infty$ is finite, then $\{a + W : a \in A_\infty\}$ forms a basis for the quotient vector space $V/W = \{v + W : v \in V\}$. (Here $V/W$ is a vector space with the operations defined by $(u + W) + (v + W) = u + v + W$ and $a(v + W) = (av) + W$; it is not difficult to check that these operations satisfy the axioms of a vector space.) Since $\{b + W : b \in B_\infty\}$ forms a basis for $V/W$, it follows from the finite form of the Dimension Theorem that $|A_\infty| = |B_\infty|$. Let $W'$ be the span of $W \cup A_\infty$ and observe that $W'$ is also the span of $W \cup B_\infty$. If we let $f : A_\infty \to B_\infty$ be any bijection, then there is a unique linear transformation $T'$ from $W'$ to $W'$, which agrees with $T$ on $W$ and which agrees with $f$ on $A_\infty$. But this means that $T' : W' \to W'$ is an element of $\mathbb{P}$ strictly above $T$, a contradiction to our assumption that $T$ was maximal.

Now suppose that $A_\infty$ is infinite. Observe that $B_\infty$ is also infinite and that both are countable. Let $f : A_\infty \to B_\infty$ be a bijection and extend $T$ to a linear transformation $T'$, which is defined on the span $W'$ of $W \cup A_\infty$ and satisfies $T'(a) = f(a)$ for each $a$ in $A_\infty$. Observe that the range of $T$ is the span of $W \cup B_\infty$, which is, by construction, the span of $W \cup A_\infty$. This again contradicts our assumption that $T$ was a maximal element of $\mathbb{P}$. Hence the domain of $T$ was all of $V$ and therefore $T$ defines a bijection between $A$ and $B$. This completes the proof. ■

Martin Kassabov has pointed out that there is a vector space $V$ with bases $A$ and $B$, such that there are no nontrivial elements of $\mathbb{P}$ that are finite dimensional linear transformations. For instance, take $V$ to be the vector space spanned by vectors $A = \{a_n : n \in \mathbb{N}\}$. Define

$$b_{2n-1} = a_{2n-1} + a_{2n} + a_{2n+1} \quad \text{and} \quad b_{2n} = a_{2n} + a_{2n+1}.$$

Notice that we then have that

$$a_{2n-1} = b_{2n-1} - b_{2n},$$
$$a_{2n} = b_{2n} - b_{2n+1} + b_{2n+2}$$

and in particular, $B = \{b_n : n \in \mathbb{N}\}$ is also a basis. Furthermore, in order to express $a_n$ in the basis $B$, we must use a coordinate greater than $n$ (and similarly for $b_n$ and $A$). That is, $a_n$ is not in the span of $\{b_i : i \le n\}$ and $b_n$ is not in the span of $\{a_i : i \le n\}$. It follows that the only element of $\mathbb{P}$ with a finite dimensional domain is the trivial element.

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—Submitted by Scott Chapman, Editor