ON A NECESSARY AND SUFFICIENT CONDITION FOR RIEMANN INTEGRABILITY

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Let D be the set of points of discontinuity of $f: I \to R$ where I is any interval of the real line R. Let $L = \{x : f$ has a left-hand limit at $x\}$.

La Vita [1] proved that if S is a closed subset of L then $D \cap S$ is countable. He then made use of this result to prove that f is continuous a.e. if and only if f has a left-hand limit a.e. Thus Lebesgue's well-known necessary and sufficient condition for Riemann integrability of a bounded function on a finite interval can be replaced by the "weaker" condition that f have a left-hand limit a.e.

It is the purpose of this note to present a much shorter and simpler proof than that given in [1] of the following stronger result, from which the equivalence of the two conditions for Riemann integrability follows immediately.

THEOREM. *$D \cap L$ is countable.*

Proof. Let $D_n = \{x : \operatorname{osc}(f, x) > 1/n\}$, $n = 1, 2, 3, \ldots$, where

$$\operatorname{osc}(f, a) = \lim_{\delta \to 0+} (\sup\{f(x) : |x - a| < \delta\} - \inf\{f(x) : |x - a| < \delta\}).$$

Since $D = \bigcup_{n=1}^{\infty} D_n$, and therefore $D \cap L = \bigcup_{n=1}^{\infty} D_n \cap L$, we need only prove that $D_n \cap L$ is countable for each n.

Suppose $x_0 \in D_n \cap L$. Since $x_0 \in L$, there exists a $\delta > 0$ such that $|f(x) - f(x_0^-)| < 1/2n$ for all $x \in (x_0 - \delta, x_0)$. Hence

$$|f(x_1) - f(x_2)| < 1/n \quad \text{for} \quad x_1, x_2 \in (x_0 - \delta, x_0).$$

It follows that if $x \in (x_0 - \delta, x_0)$ then $\operatorname{osc}(f, x) \leqq 1/n$, so that $x \notin D_n$. Thus any point of $D_n \cap L$ is the right endpoint of an open interval which contains no point of $D_n \cap L$. Since these open intervals are clearly disjoint, and hence form a countable set, it follows that $D_n \cap L$ is countable, and the theorem is proved.

References

1. J. A. La Vita, A necessary and sufficient condition for Riemann integration, this MONTHLY, 71 (1964) 193–196.

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A BOUNDED DERIVATIVE WHICH IS NOT RIEMANN INTEGRABLE

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We give a very simple example of a bounded derivative which is not Riemann integrable. Let $f: [0, 1] \to R$ be defined as follows. Let $G \subset [0, 1]$ be a dense open set which is the union of pairwise disjoint open intervals, $\{I_n\}$, the sum of whose lengths is $\frac{1}{2}$. For each n, let $J_n \subset I_n$ be a closed interval in the center of I_n such that the lengths satisfy $l(J_n) = [l(I_n)]^2$. For each n, define f on J_n to be continuous, 1 at the center, 0 at the end points and always between 0 and 1. Define f to be 0 everywhere else.

The function f is not Riemann integrable. For if π is any partitioning of $[0, 1]$ the intervals of π in which the oscillation of f is 1 have length sum exceeding $\frac{1}{2}$, so that

$$\overline{\int} f - \underline{\int} f \geqq \frac{1}{2}.$$

That f is a derivative follows from the fact that it is bounded and approximately continuous so that it is the derivative of its indefinite Lebesgue integral $F(x) = \int_0^x f(t)dt$. It is of interest that this may be shown without the use of the Lebesgue integral. The function F may be obtained as an improper integral by letting

$$F(x) = \sum_{n=1}^{\infty} \int_{K_n} f(t)dt,$$

where $K_n = J_n \cap [0, x]$, $n = 1, 2, \ldots$. Let $I \subset [0, 1]$ be an interval which meets the complement of G, and let n be such that $I \cap J_n \neq \emptyset$. Let $S_n = l(I_n)$. Since $S_n \leqq \frac{1}{2}$, it follows that $l(I \cap I_n) \geqq \frac{1}{2}(S_n - S_n^2) \geqq \frac{1}{4} S_n$. Then $l(I \cap J_n) \leqq l(J_n) = S_n^2 \leqq 16\{l(I \cap I_n\}^2$. If $N = \{n : I \cap J_n \neq \emptyset\}$, then

$$\sum_{n \in N} l(I \cap J_n) \leqq \sum_{n \in N} 16\{l(I \cap I_n)\}^2 \leqq 16\{l(I)\}^2.$$

For $x \notin G$ and $y \neq x$, we have $\int_x^y f(t)dt \leqq 16(y - x)^2$, whence $F'(x) = 0$. That $F'(x) = f(x)$ on G is obvious.

For comparison see [**1**], [**2**], and [**3**].

References

1. C. Goffman, Real Functions, Rinehart, New York, 1960, p. 210–11.
2. E. W. Hobson, The Theory of Functions of a Real Variable and the Theory of Fourier Series, Vol. 2, Cambridge, p. 412–421.
3. I. P. Natanson, Theory of Functions of a Real Variable, vol. 1, Ungar, New York, 1960, p. 133.

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HOMOLOGICAL DOTS

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In this note, I would like to introduce a new mathematical game. I originally called it "Commutative Diagrams and Exact Sequences," a name which accurately reflects its essence, but for reasons of euphony, as well as its resemblance to the children's game of Dots, I have changed the name to the one above.

The rules are as follows:

1. The game is played on a 5×5 lattice.
2. The first move consists of writing down an arbitrary finitely generated abelian group at some lattice point.
3. Any subsequent move consists of writing down an arbitrary finitely generated abelian group on a vacant lattice point which is orthogonally adjacent to an already occupied lattice point, together with maps to/from the adjacent group(s), subject to the following conditions:

(a) All arrows are to the right or down.
(b) All horizontal or vertical sequences must be exact.
(c) All squares must commute.

4. Scoring: Suppose a player makes a move which makes it impossible to legally fill some other