TRICKS OR TREATS WITH THE HILBERT MATRIX

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This article is intended to be a partial account on everything you always wanted to know about the *Hilbert matrix*—namely, the infinite Hilbert matrix A or the finite order $n \times n$ Hilbert matrix A_n:

$$A = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & & \\ \frac{1}{3} & \frac{1}{4} & & & \\ \frac{1}{4} & & & & \\ \vdots & & & & \end{bmatrix}, \quad A_n = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{3} & & & \\ \frac{1}{3} & & & & \\ \vdots & & & & \vdots \\ \frac{1}{n} & & & \cdots & \frac{1}{2n-1} \end{bmatrix}$$

It is natural to ask: What are the natural consequences from such a natural assemblage of (the reciprocals of) natural numbers? Hence, here arise ten concrete *problems* (instead of theorems), aimed to reveal the "heart of mathematics" in a broad sense. All solutions provided in this article are elementary (requiring only the rudiments of linear algebra or simple operator theory without the spectral theorem). However, many tricks or treats associated with the Hilbert matrix may seem rather frightening or fascinating.

The Hilbert matrix has played a prominent role in the structure theory of several branches of mathematics. Indeed, it serves as one of the most vivid examples for many unusual aspects (as well as usual aspects, of course) in operator theory (see e.g., [**1**], [4], [7]). Undoubtedly, the infinite Hilbert matrix stands out from other single operators, illustrating many soft results by hard analysis, discrete features of continuous phenomena, and subtle effects through intricate combinatorics.

This pseudo-expository article is designed to exhibit some of these phenomena. The main body consists of ten problems on the Hilbert matrix, showing various aspects of concrete and discrete nature. The majority of these problems are apparently new, but many of their variants and underpinnings have already appeared in the literature. The second half of the article is devoted to solutions and additional notes and references. Much effort has been made to seek the most elementary solutions instead of the best answers to the problems. Interested readers will probably want to find other solutions that are less elementary but provide still more information.

It may be appropriate to fix some notations and terms here. The term "matrix" refers to a finite-order or infinite square matrix with complex entries (or real entries, without loss of generality). The letter I stands for the identity matrix with ones along the main diagonal and zeros elsewhere. A matrix $T = [\tau_{ij}]_{i,j}$ is said to be *positive*, in notation $T \geqslant 0$, if $\Sigma \tau_{ij} \bar{x}_i x_j \geqslant 0$ for all complex k-tuples $(x_1, x_2, \ldots, x_k)$ with $k \geqslant 0$. The notation $\|T\|$ refers to the *Hilbert-space-operator norm*; thus if an operator T: $l^2 \to l^2$ has the matrix representation $[\tau_{ij}]$, then the operator norm of T (that is, the operator norm of $[\tau_{ij}]$) is

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$$\|T\| = \sup\left\{\left(\sum_i\left|\sum_j \tau_{ij}\alpha_j\right|^2\right)^{1/2} : \sum|\alpha_j|^2 \leqslant 1\right\}.$$

There is a useful elementary fact: if $T = [\tau_{ij}]$ and $S = [\sigma_{ij}]$ with $0 \leqslant \sigma_{ij} \leqslant \tau_{ij}$ for all i and j, then $\|S\| \leqslant \|T\|$.

I. Invertibility. By computation, $A_1^{-1} = 1$,

$$A_2^{-1} = \begin{bmatrix} 4 & -6 \\ -6 & 12 \end{bmatrix}, \quad A_3^{-1} = \begin{bmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{bmatrix}$$

Thus, the general case becomes irresistible.

PROBLEM: *Write out the inverse of the $n \times n$ Hilbert matrix A_n explicitly.*

It may be instructive to expose what you were afraid to ask:

(a) While each entry of A_n is the reciprocal of an integer, what sort of coincidence is it if A_n^{-1} has integer entries?

(b) In particular, why is det A_n the reciprocal of an integer?

(c) Since each entry of A_n is a positive real number, does it follow that A_n^{-1} is of the form $[\beta_{ij}]$ with $(-1)^{i+j}\beta_{ij} \geqslant 0$ for all i, j?

(d) While A_n is a *Hankel matrix* (i.e., $A_n = [\alpha_{ij}]$ with $\alpha_{ij} = \alpha_{pq}$ whenever $i + j = p + q$), should there be any pattern for A_n^{-1} at all?

II. Formal Inverse. With all the tractable A_n^{-1} already in hand, can A^{-1} be still far beyond the grasp?

In this section, the notation Σ is used only for sequential summation—absolute convergence is not required. There are three intrinsic properties pertinent to each infinite matrix $T = [\tau_{ij}]$: T is said to be *formally one-to-one* if the trivial sequence is the only sequence (α_j) satisfying $\Sigma_j \tau_{ij}\alpha_j = 0$ for each i; T is said to be *formally onto* if for each sequence (α_j) there is a sequence (β_j) such that $\Sigma_j \tau_{ij}\alpha_j = \beta_i$ for each i; $S = [\sigma_{ij}]$ is called a *formal inverse* of T if the formal products TS and ST are defined and equal to I; that is,

$$\sum_k \tau_{ik}\sigma_{kj} = \sum_k \sigma_{ik}\tau_{kj} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Since the formal multiplication of infinite matrices is nonassociative, there is no actual correlation among these three properties.

PROBLEM: (1) *Is the infinite Hilbert matrix A formally one-to-one?*

(2) *Is A formally onto?*

(3) *Does A have a formal inverse?*

III. Strong Positivity. If $T = [\tau_{ij}]$ is a matrix with $\tau_{ij} \geqslant 0$ and r is a real number, then let $T^{[r]}$ denote the matrix $[\tau_{ij}^r]$.

It is worthwhile to praise the $n \times n$ Hilbert matrix A_n for all its positivity in the highest.

PROBLEM: *Prove that $A_n^{[r]} \geqslant 0$ for each real number $r \in (0, \infty)$.*

Note that in general, "$T \geqslant 0$" need not imply "$T^{[r]} \geqslant 0$"; example:

$$T = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \geqslant 0, \quad \text{but } T^{[1/2]} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2^{1/2} & 1 \\ 0 & 1 & 1 \end{bmatrix} \ngeqslant 0.$$

IV. Factorization. In view of the positivity of the infinite Hilbert matrix, Paul Halmos (oral communication) has asked: Is it possible to write out explicitly $A = B^2$ with $B \geqslant 0$? This question looks very frightening—apparently, it may involve finding the spectral decomposition of A. Still, there is a more tractable problem.

PROBLEM: *Write out explicitly* $A = BB^*$ *where* B *is lower-triangular* (i.e., $B = [\beta_{ij}]_{1 \leqslant i, j < \infty}$ with $\beta_{ij} = 0$ whenever $i < j$). (Note that such B is unique if, in addition, all diagonal entries of B are positive real numbers.)

There are some merits for the triangular factorization:

(a) Let $B_n = [\beta_{ij}]_{1 \leqslant i, j \leqslant n}$ be the $n \times n$ upper-left corner of B. Then it follows $A_n = B_n B_n^*$. Thus an induction to seek B is admissible.

(b) Furthermore, $\det A_n = |\det B_n|^2$ is completely determined by the diagonal entries of B. (Cf. remark (b) following the problem in Section I.)

(c) Because of its very tractable structure, each lower-triangular infinite matrix with nonzero diagonal entries always admits a formal inverse which is also lower-triangular. Thus let C be the formal inverse of B. Then the abstract expression $C^*C = A^{-1}$ makes sense in the following manner: If $Av = w$, then $C^*(Cw) = v$—the formal matrix manipulations for Cw and $C^*(Cw)$ can be carried out even though the formal matrix product C^*C is not defined.

V. Companion. A loyal companion of the Hilbert matrix is the matrix

$$L = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{3} & \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \\ \vdots & & & \end{bmatrix} \quad \text{or } L_n = \begin{bmatrix} 1 & \frac{1}{2} & \cdots & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{2} & & \\ \vdots & & & \vdots \\ \frac{1}{n} & & \cdots & \frac{1}{n} \end{bmatrix}$$

whose entries run in a reversed-L-shaped pattern.

Notice that L (or L_n) is an assemblage of reciprocals of positive integers, too. Aside fr(external resemblance, L (or L_n) also enjoys all those delightful properties as A (or A_n) does in the preceding problems. Notably, L admits a triangular factorization $L = CC^*$ where C is the infinite Cesàro matrix,

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \\ \vdots & & & & \end{bmatrix}.$$

It is very pleasant to call upon the companion L for the proof that $A : l^2 \to l^2$ is a bounded operator.

PROBLÉM: (1) *Observe that* $I - (I - C)(I - C)^* \geqslant 0$.

(2) *Prove that* $\|A\| \leqslant \|L\| \leqslant 4$.

Apparently, this problem provides the quickest elementary way to show that A is a bounded operator.

VI. Heredity. The immediate offspring of the infinite Hilbert matrix A is

$$A' = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\ \frac{1}{3} & \frac{1}{4} & & \\ \frac{1}{4} & & & \\ \vdots & & & \end{bmatrix}$$

which has shared the name of *Hilbert matrix* in literature.

Undoubtedly, A' deserves the title of Hilbert matrix because A' indeed carries on all heredity.

PROBLEM: *Show that A and A' are unitarily equivalent* (i.e., there exists a unitary operator U such that $A = U^*A'U$).

It is apparently surprising to see that (a) A' is entrywise strictly smaller than A, and (b) A' is a proper part of A—the first row (or the first column) of A has been erased, yet it turns out that A' is unitarily equivalent to A.

VII. Compact Perturbation. Many offspring of the infinite Hilbert matrix A can be obtained by simply erasing the first p_1 rows and the first p_2 columns from A. Thus, there are infinitely many Hankel matrices

$$A(q) = \begin{bmatrix} \frac{1}{q+1} & \frac{1}{q+2} & \frac{1}{q+3} & \cdots \\ \frac{1}{q+2} & \frac{1}{q+3} & & \\ \frac{1}{q+3} & & & \\ \vdots & & & \end{bmatrix}$$

with $q = p_1 + p_2$ being any nonnegative integer. In particular, $A(0) = A$, $A(1) = A'$.

It is a pleasure to report below that all these offspring are *essentially the same* (i.e., the difference of any two of them is a compact operator). Let $T_n = [\tau_{ij}]_{1 \leqslant i, j \leqslant n}$ be the $n \times n$ upper-left corner matrix of $T = [\tau_{ij}]_{1 \leqslant i, j < \infty}$; T_n can be regarded as an infinite matrix $[\alpha_{ij}]_{1 \leqslant i, j < \infty}$ with

$$\alpha_{ij} = \begin{cases} \tau_{ij} & \text{if } i \leqslant n \text{ and } j \leqslant n, \\ 0 & \text{elsewhere.} \end{cases}$$

By one of many equivalent definitions, T is said to be *compact* if $\lim_{n\to\infty}\|T - T_n\| = 0$; T is said to be of *trace-class* if the sum of the main diagonal entries of $(T^*T)^{1/2}$ is finite. It is well known that

$$\{\text{trace-class operators}\} \subset \{\text{compact operators}\} \subset \{\text{bounded operators}\}$$

where $\subset$ stands for a proper inclusion as a linear submanifold.

PROBLEM: (1) *Show that the infinite Hilbert matrix A is not a compact operator.*
(2) *Show that $A - A(q)$ is a trace-class operator.*

VIII. The Euler Recipe. There are many possible ways to make π; but it is striking to get π from the magnificent formula of Euler

$$\pi^2/6 = 1 + 1/2^2 + 1/3^2 + 1/4^2 + \cdots.$$

A countable ordered set Γ is called an *Euler vector* if Γ as an unordered set is the same as $\{0\} \cup \{1/k\colon k \in \mathbb{Z} \setminus \{0\}\}$, and furthermore each nonzero element $1/k$ occurs three times exactly in Γ. By Euler's formula, it follows immediately that each Euler vector is of norm π in the l^2-norm.

Next, recall that a matrix T_1 is a *dilation* of a matrix T_0 if T_1 can be written in the form $\begin{bmatrix} X & Y \\ Z & T_0 \end{bmatrix}$. In other words, if $\mathcal{H}_0$ is a subspace of a Hilbert space $\mathcal{H}_1$ and P is the orthogonal projection from $\mathcal{H}_1$ onto $\mathcal{H}_0$, and if $T_1\colon \mathcal{H}_1 \to \mathcal{H}_1$, $T_0\colon \mathcal{H}_0 \to \mathcal{H}_0$ are operators satisfying $PT_1|_{\mathcal{H}_0} = T_0$, then T_1 is called a *dilation* of T_0.

PROBLEM: *Dilate the infinite Hilbert matrix A (or its immediate offspring A') to a real symmetric matrix T such that each column-vector of T is an Euler vector, and furthermore any two column-vec-*

tors of T are orthogonal to each other.

Consequently, $T^2 = \pi^2 I$ and $\|T\| = \pi$; therefore $\|A\| \leqslant \|T\| = \pi$.

IX. π Again. A close relative of the $n \times n$ Hilbert matrix A_n is

$$Z_n = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{3} & & & 0 \\ \frac{1}{3} & & & & \\ \vdots & & & & \vdots \\ \frac{1}{n} & 0 & & \cdots & 0 \end{bmatrix},$$

a finite-order Hankel matrix with all zeros below the main cross-diagonal. Thus "by analogy," is there a corresponding relative of the infinite Hilbert matrix A? *Answer*: A itself (plausible but not convincing?); in other words, the infinite-dimensional counterpart of Z_n ought to be A (still controversial?).

Anyhow, there is an interesting connection between the equalities $\pi = 2$ Arcsin 1 and $\|A\| = \lim_{n\to\infty}\|Z_n\|$ (instead of $\|A\| = \lim_{n\to\infty}\|A_n\|$).

PROBLEM: (1) *Show that*

$$\pi = \lim_{k\to\infty} \left\{(1k)^{-1/2} + (2(k-1))^{-1/2} + \cdots + (k1)^{-1/2}\right\}.$$

(2) *Use* (1) *or some others to show that* $\lim_{n\to\infty}\|Z_n\| \geqslant \pi$.

The combination of the results of Problems in Sections VIII and IX brings home the delightful fact $\|A\| = \pi$.

X. Two Natural Dilations. The Hilbert matrix A is a "one-way-infinite Hankel matrix." In contrast, there is a family of "two-way-infinite Hankel matrices" of the form

$$\left[\begin{array}{cc|cc} & \vdots & & \cdot^{\cdot^{\cdot}} \\ & \alpha_{-2} & \alpha_{-1} & \alpha_0 \\ \alpha_{-2} & \alpha_{-1} & \alpha_0 & \alpha_1 \\ \hline \alpha_{-1} & \alpha_0 & \alpha_1 & \alpha_2 \\ \alpha_0 & \alpha_1 & \alpha_2 & \\ \cdot^{\cdot^{\cdot}} & & \vdots & \end{array}\right].$$

Two of these matrices arise as the natural dilations of A: the first is denoted by $A^{\#}$ with

$$\alpha_k = \begin{cases} 0 & \text{if } k = 0, \\ 1/k & \text{if } k \text{ is a nonzero integer;} \end{cases}$$

the second is denoted by $A^{\dagger}$ with

$$\alpha_k = \begin{cases} 0 & \text{if } k \text{ is a nonpositive integer,} \\ 1/k & \text{if } k \text{ is a positive integer.} \end{cases}$$

PROBLEM: (1) *Show that* $\|A^{\#}\| = \pi$.
(2) *Show that* $\|A^{\dagger}\| = \infty$.

By (1), it follows again that $\|A\| \leqslant \|A^{\#}\| = \pi$. Apparently, this is the simplest way to show $\|A\| \leqslant \pi$. (Cf. problems in Sections V and VIII.)

The combination of (1) and (2) yields an extremely amazing fact: Although $A^{\dagger}$ is a lower-right-triangular part of $A^{\#}$, it turns out that $A^{\dagger}$ blows up while $A^{\#}$ is still bounded.

Alternatively, rewrite

$$A^{\#} = \begin{bmatrix} -A & B^* \\ B & A \end{bmatrix}, \quad A^{\dagger} = \begin{bmatrix} 0 & B_0^* \\ B_0 & A \end{bmatrix}$$

where B and B_0 are one-way-infinite Toeplitz matrices

$$B = \begin{bmatrix} 0 & -1 & -\frac{1}{2} & -\frac{1}{3} & \cdots \\ 1 & 0 & -1 & -\frac{1}{2} & \\ \frac{1}{2} & 1 & 0 & -1 & \\ \frac{1}{3} & \frac{1}{2} & 1 & 0 & \\ \vdots & & & & \ddots \end{bmatrix}, \quad B_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \\ \frac{1}{2} & 1 & 0 & 0 & \\ \frac{1}{3} & \frac{1}{2} & 1 & 0 & \\ \vdots & & & & \ddots \end{bmatrix}.$$

Then it follows that $\|B\| \leqslant \|A^{\#}\| < \infty$ and $\|B_0\| = \|A^{\dagger}\| = \infty$, yet B_0 is the lower-left-triangular part of B.

SOLUTIONS AND NOTES

I. Solution. Let

$$\alpha_{ij} = (-1)^{i+j}(i+j-1)\binom{n+i-1}{n-j}\binom{n+j-1}{n-i}\binom{i+j-2}{i-1}^2.$$

The direct but cumbersome computation leading to $[\alpha_{ij}] = A_n^{-1}$ can be justified by means of mathematical inductions (on i, on j, or on n) or some general counting principles.

Alternatively, it may be easier to apply a known fact:

$$\det\left[1/(x_i + y_j)\right] = \Pi_{j>i}\left((x_j - x_i)(y_j - y_i)\right)/\Pi_{i,j}(x_i + y_j)$$

(see [**6**, p. 92, Problem 3]). Let $x_i = i - 1$, $y_j = j$; it follows immediately that

$$\det A_n = \frac{\{1!2!\cdots(n-1)!\}^4}{1!2!\cdots(2n-1)!}.$$

Similarly, all cofactors of A_n can be computed out explicitly. Henceforth $A_n^{-1} = [\alpha_{ij}]$ follows.

Note. The answer to this problem has been known for many years. The solution provided above has already appeared in [**9**]. See [**13**] for other related computational problems.

II. Solution. (1) Some observations are needed so as to reduce tedious computation.

(a) If A annihilates a sequence $(0,\ldots,0,1,\gamma_1,\gamma_2,\ldots)$ whose leading nonzero entry is 1, then $1 < |\gamma_1| + |\gamma_2| + \cdots$.

(b) If A annihilates a sequence $(\alpha_1, \alpha_2, \alpha_3, \ldots)$, then A also annihilates $(0, \alpha_1, \alpha_2/2, \alpha_3/3, \ldots)$.

(c) If A annihilates a nontrivial sequence, then A also annihilates a nontrivial l^1-sequence.

(d) If A annihilates a sequence $(0,\ldots,0,1,\beta_1,\beta_2,\ldots)$ whose leading nonzero entry is 1 at the pth position, then A also annihilates a sequence $(0,\ldots,0,0,1,\gamma_1,\gamma_2,\ldots)$ whose leading nonzero entry is 1 at the $(p+1)$st position and $|\gamma_j| \leqslant p|\beta_j|/(p+1)$ for all j.

Statement (a) is obvious. Statement (b) follows from the computation

$$0 = \sum_{j=1}^{\infty} \alpha_j/(i+j-1) \quad \text{for each } i = 1,2,3,\ldots$$

$$\Rightarrow 0 = \frac{1}{i}\left[\sum_{j=1}^{\infty} \alpha_j/j - \sum_{j=1}^{\infty} \alpha_j/(i+j)\right]$$

$$= \sum_{j=1}^{\infty} \alpha_j/(j(i+j)) = \sum_{j=2}^{\infty} \alpha_{j-1}/((j-1)(i+j-1)).$$

To get (c), apply (b) three times; thus if A annihilates a nontrivial sequence $(\alpha_1, \alpha_2, \alpha_3, \dots)$, then A also annihilates

$$(0,0,0,\alpha_1/(1\cdot 2\cdot 3),\dots,\alpha_j/(j(j+1)(j+2)),\dots)$$

which is an l^1-sequence in virtue of the fact $\Sigma\alpha_j/j = 0$ (from the assumption). Statement (d) is an immediate consequence of (b).

Now suppose that A is not formally one-to-one. Then by (c), we may assume that A annihilates a sequence $(0,\dots,0,1,\beta_1,\beta_2,\dots)$ whose leading coefficient is 1 at the pth position, and $|\beta_1| + |\beta_2| + \cdots < \infty$. Application of Statement (d) n times yields that A also annihilates a sequence $(0,\dots,0,1,\gamma_1,\gamma_2,\dots)$ whose leading entry is 1 at the $(p+n)$th position and $|\gamma_j| \leqslant p|\beta_j|/(p+n)$. Thus

$$|\gamma_1| + |\gamma_2| + \cdots \leqslant p(|\beta_1| + |\beta_2| + \cdots)/(p+n) \leqslant 1$$

when n is sufficiently large. But by (a), this leads to contradiction. Therefore A must be formally one-to-one.

(2) and (3). Observe that whenever A maps a sequence $(\alpha_1, \alpha_2, \alpha_3, \dots)$ to $(1,0,0,\dots)$, then A will automatically annihilate $(0, \alpha_1, \alpha_2, \alpha_3, \dots)$. Since A is formally one-to-one, it follows that the sequence $(1,0,0,\dots)$ is not in the formal range of A. This, of course, shows that A does not have a formal inverse.

Note: It is well known that $A\colon l^2 \to l^2$ is one-to-one. (See e.g. [**5**, Theorem 1, pp. 703–4] for the usual proof.) On the other hand, M. Rosenblum [**8**] has shown that for each complex number λ with $\operatorname{Re}\lambda > 0$, there is a sequence $v = (\alpha_n)$ with $\Sigma|\alpha_n|^2 = \infty$ satisfying the formal equality $Av = \lambda v$.

III. Solution. If x and r are real numbers and $r > 0, |x| < 1$, then by the binomial expansion,

$$(1-x)^{-r} = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \cdots$$

where $\alpha_0 = 1$, $\alpha_1 = r$, and each α_k is a positive real number. Thus

$$i + j - 1 = ij(1-x) \quad \text{with} \quad x = (i-1)(j-1)/(ij) < 1,$$

and

$$(i+j-1)^{-r} = i^{-r}j^{-r}(1-x)^{-r} = \sum_k \alpha_k (i-1)^k (j-1)^k/(ij)^{r+k}.$$

Notice that $[\alpha_k(i-1)^k(j-1)^k/(ij)^{r+k}]_{i,j}$ is a rank 1 positive matrix. Therefore $A_n^{[r]} = [(i+j-1)^{-r}]_{i,j}$, as a sum of positive matrices, is also positive.

Note: In particular, let $r = 1$, then $\alpha_k = 1$ for each k; thus the paragraph above provides a simple "discrete" proof for the well-known fact $A_n \geqslant 0$ (see e.g., [**1**, Section (8)] for the usual proof).

IV. Solution. As indicated in the text, the construction of B by the mathematical induction is admissible. Thus $A = BB^*$ where $B = [\beta_{ij}]$ with

$$\beta_{ij} = \begin{cases} \dfrac{\sqrt{2j-1}\,((i-1)!)^2}{(i+j-1)!(i-j)!} & \text{if } i \geqslant j, \\ 0 & \text{if } i < j. \end{cases}$$

Notes. (1) The matrix B can also be constructed as follows. Regard l^2 as $L^2[0,1]$ with the fixed orthonormal basis $\{f_1, f_2, \dots\}$ where $f_n(x)$ is a polynomial of degree $n-1$ carrying a positive coefficient in the x^{n-1} term. Let T: $L^2[0,1] \to L^2[0,1]$ be the continuous linear map sending each f_n to x^{n-1}. Henceforth, it can be verified that B^* is just the matrix representation for T.

(2) For a graceful expression, it may be better to write $A = EXDX^*E$ where

$$E = \begin{bmatrix} 1 & & & & \\ & \frac{1}{2} & & & \\ & & \frac{1}{3} & & \\ & & & \frac{1}{4} & \\ & & & & \ddots \end{bmatrix}, \quad D = \begin{bmatrix} 1 & & & & \\ & 3 & & & \\ & & 5 & & \\ & & & 7 & \\ & & & & \ddots \end{bmatrix},$$

$$X = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 1/3 & 0 & 0 & \\ 1 & 2/4 & 1\cdot 2/4\cdot 5 & 0 & \\ 1 & 3/5 & 2\cdot 3/5\cdot 6 & 1\cdot 2\cdot 3/5\cdot 6\cdot 7 & \\ \vdots & & & & \ddots \end{bmatrix}.$$

Although X and D are unbounded operators: $l^2 \to l^2$, there is, however, no difficulty to carry out the formal matrix computation $B = EXD^{1/2}$, $A = BB^*$.

(3) B has a formal inverse $[\gamma_{ij}]$ where

$$\gamma_{ij} = \begin{cases} (-1)^{i+j}\sqrt{2i-1}\binom{i+j-2}{j-1}\binom{i-1}{j-1} & \text{if } i \geqslant j, \\ 0 & \text{if } i < j. \end{cases}$$

This result has been noticed by J. Todd [**12**].

V. Solution. (1) is plain since $I - (I - C)(I - C^*)$ is a diagonal operator with positive diagonal entries $1, 1/2, 1/3, \dots$.

(2) Here the equality $\|TT^*\| = \|T\|^2$ for the Hilbert-space-operator norm will be used twice. First from (1),

$$\|I - C\|^2 = \|(I - C)(I - C)^*\| \leqslant 1;$$

i.e., $\|I - C\| \leqslant 1, \|C\| \leqslant 2$. Next,

$$\|L\| = \|CC^*\| = \|C\|^2 \leqslant 4.$$

Since each entry of A is a positive real number smaller than or equal to the corresponding entry of L, it follows that $\|A\| \leqslant \|L\| \leqslant 4$ as desired.

Note: This very simple proof that $\|C\| \leqslant 2$ has appeared in [**2**, pp. 128–129].

VI. Solution. Let C be the infinite Cesáro matrix as mentioned in Section V. It is easy to see that both C: $l^2 \to l^2$ and C^*: $l^2 \to l^2$ are one-to-one (thus, C is of dense range). Next, a direct computation yields $CA = A'C$. The rest of the proof is the restatement of the following lemma which is known to many operator-theorists.

LEMMA: *Suppose T, S, and C are bounded operators satisfying $CT = SC$. If both T and S are hermitian and if C is one-to-one and of dense range, then T and S are unitarily equivalent.*

Proof. By the polar decomposition, there is a unitary U and a positive P such that $C = UP$ and P is one-to-one and of dense range (see e.g. [**3**, p. 169]). Thus

$$P^2T = PU^*UPT = C^*CT = C^*SC = (SC)^*C = (CT)^*C = TC^*C = TPU^*UP = TP^2,$$

and consequently $PT = TP$. Therefore,

$$UTP = UPT = CT = SC = SUP.$$

Since P is one-to-one and of dense range, it follows $UT = SU$; i.e., $T = U^*SU$ as desired.

VII. Solution. (1) Clearly, $A - A_n$ contains an $n \times n$ submatrix

$$Y = \begin{bmatrix} \frac{1}{2n+1} & \frac{1}{2n+2} & \cdots & \frac{1}{3n} \\ \frac{1}{2n+2} & & & \\ \vdots & & & \vdots \\ \frac{1}{3n} & & \cdots & \frac{1}{4n-1} \end{bmatrix}.$$

Let Q be the $n \times n$ matrix where every entry is $1/n$; then Q is an orthogonal projection (i.e., $Q = Q^* = Q^2$) and thus of norm 1. Since Y is entrywise larger than $Q/4$, it follows $\|A - A_n\| \geqslant \|Y\| \geqslant \|Q/4\| = 1/4$. This proves that A is not compact.

(2) First observe that $A(q) \geqslant 0$. This is an immediate consequence of the fact that A (or A') is a dilation of $A(q)$ if q is an even (or odd) integer. Next, use the notion of *Schur product*: if $T = [\tau_{ij}]$ and $S = [\sigma_{ij}]$, then the Schur product $T * S = [\tau_{ij}\sigma_{ij}]$; it is an elementary fact that if $T \geqslant 0$ and $S \geqslant 0$ then $T * S \geqslant 0$. Henceforth,

$$A - A(q) = \left[\frac{1}{i+j-1} - \frac{1}{q+i+j-1}\right]_{ij}$$
$$= \left[\frac{q}{(i+j-1)(q+i+j-1)}\right]_{ij} = qA * A(q).$$

Since $A \geqslant 0$ and $A(q) \geqslant 0$, it follows that $A * A(q) \geqslant 0$ and thus $A - A(q) \geqslant 0$. Finally, observe that the sum of diagonal entries of the positive operator $A - A(q)$ is

$$\sum_j q/((2j-1)(q+2j-1)) < \infty.$$

Therefore $A - A(q)$ is a trace-class operator.

Note. It is well known (see e.g. [**1**, Section 8]) that A is not compact. It is even easier to see that $A - A(q)$ is a *Hilbert-Schmidt* operator (i.e., the sum of squares of entries of $A - A(q)$ is finite), as noted in [**1**, Section 8].

VIII. Solution. First define an auxiliary function $\Phi\colon \mathbb{Z}^2 \to \mathbb{R}$ by

$$\Phi(k,l) = \begin{cases} 1/k & \text{if } k \neq 0, l = 0 \\ 1/l & \text{if } k = 0, l \neq 0 \\ -1/k & \text{if } k = l \neq 0 \\ 0 & \text{elsewhere.} \end{cases}$$

It follows that

(a) Φ, regarded as an ordered set, is an Euler vector,

(b) if (k_0, l_0) is a fixed pair of integers different from $(0,0)$, then

$$\sum_{k,l} \Phi(k,l)\Phi(k+k_0, l+l_0) = 0.$$

Next, let $\mathcal{H}_0$ be the Hilbert space l^2 with the orthonormal basis $\{f_1, f_2, \dots\}$ and let $\mathcal{H}_1$ be $l^2(\mathbb{Z}^2)$ with the orthonormal basis $\{e_\mu\colon \mu \in \mathbb{Z}^2\}$; imbed $\mathcal{H}_0$ into $\mathcal{H}_1$ by identifying each f_j with e_μ

where $\mu = (j,0)$. Thus the infinite Hilbert matrix A (or A') stands for an operator: $\mathcal{H}_0 \to \mathcal{H}_0$ while two matrices $T = [\tau_{\mu\nu}]$, $S = [\sigma_{\mu\nu}]$ ($\mu, \nu \in \mathbb{Z}^2$) to be constructed below are operators: $\mathcal{H}_1 \to \mathcal{H}_1$. Now, define

$$\left.\begin{array}{l}\tau_{\mu\nu} = \Phi(k-1,l)\\ \sigma_{\mu\nu} = \Phi(k,l)\end{array}\right. \quad \text{if } \mu + \nu = (k,l) \in \mathbb{Z}^2.$$

Consequently, if i and j are positive integers and $\mu = (i,0)$, $\nu = (j,0)$, then $\tau_{\mu\nu} = 1/(i+j-1)$, $\sigma_{\mu\nu} = 1/(i+j)$; i.e., T is a dilation of A, and S is a dilation of A'. From (a) and (b), it follows that each column vector of T—i.e., $(\tau_{\mu\nu})_\mu$ for each fixed ν—is an Euler vector and any two column vectors of T are orthogonal (the similar result holds for S).

Note. Alternatively, the operators T and S in the solution above can be constructed as follows. Let $\mathbb{T}$ be the unit circle with the normalized Lebesgue measure μ, let $\mathcal{H} = L^2(\mathbb{T} \times \mathbb{T}, \mu \times \mu)$, and let M: $\mathcal{H} \to \mathcal{H}$ be an operator defined by

$$g(z,w) \overset{M}{\mapsto} h(z,w) = \begin{cases} i\pi g(\bar{z},\bar{w}) & \text{if } \operatorname{Arg} z + \operatorname{Arg} w \leqslant 2\pi, \\ -i\pi g(\bar{z},\bar{w}) & \text{if } \operatorname{Arg} z + \operatorname{Arg} w > 2\pi, \end{cases}$$

subject to the constraint $2\pi > \operatorname{Arg}\, z \geqslant 0, 2\pi > \operatorname{Arg}\, w \geqslant 0$. Then it follows immediately that $M = M^*$, $M^2 = \pi^2 I$. Moreover, there is a natural way (respectively, a quasi-natural way) to identify $l^2(\mathbb{Z}^2)$ with $L^2(\mathbb{T} \times \mathbb{T})$: let $e_\mu = z^k w^l$ (resp., $e_\mu = z^{k-1}w^l$) whenever $\mu = (k,l) \in \mathbb{Z}^2$. Henceforth, it can be verified that M has the matrix expression $[\sigma_{\mu\nu}]$ (resp., $[\tau_{\mu\nu}]$) as in the solution above.

IX. Solution. (1) $\Sigma_{j=0}^{k-1}(j(k-j))^{-1/2} = \frac{1}{k}\Sigma_{j=0}^{k-1}\left(\frac{j}{k}\left(1-\frac{j}{k}\right)\right)^{-1/2}$ is a Riemann sum for the function $(x(1-x))^{-1/2}$ on $(0,1)$.

$$\therefore \lim_{k\to\infty} \sum_{j=0}^{k-1} (j(k-j))^{-1/2} = \int_0^1 (x(1-x))^{-1/2} = 2 \operatorname{Arcsin} 1 = \pi.$$

(2) Let v be the column vector $(1, 2^{-1/2}, \ldots n^{-1/2})$. Then

$$\|Z_n\| \geqslant (Z_n v, v)/\|v\|^2 = \left(\sum_{k=1}^n a_k/k\right) \Big/ \left(\sum_{k=1}^n 1/k\right)$$

where $a_k = \Sigma_{j=0}^{k-1}(j(k-j))^{-1/2}$. Since $\lim_{k\to\infty} a_n = \pi$ (from (1)), it follows by general computation $\lim_{n\to\infty}(\Sigma_{k=1}^n a_k/k)/(\Sigma_{k=1}^n 1/k) = \lim_{k\to\infty} a_k = \pi$; thus $\lim_{n\to\infty}\|Z_n\| \geqslant \pi$ as desired.

Note. The main ingredient of the proof above has appeared in [**11**]. There are other known proofs of $\|A\| = \pi$ in the literature (see e.g. [**4**, Chapter 9 and Appendix III] [**10**, Chapter 9, p. 101]).

X. Lemma. *Let T and H be two-way-infinite matrices of the form*

$$T = \left[\begin{array}{cccc|cccc} \ddots & & & \vdots & & & & \\ & & \alpha_0 & \alpha_1 & \alpha_2 & & & \\ & & \alpha_{-1} & \alpha_0 & \alpha_1 & \alpha_2 & & \\ \hline & & \alpha_{-2} & \alpha_{-1} & \alpha_0 & \alpha_1 & & \\ & & & \alpha_{-2} & \alpha_{-1} & \alpha_0 & & \\ & & & & \vdots & & & \ddots \end{array}\right], \quad H = \left[\begin{array}{cccc|cccc} & & & \vdots & & & & \cdot^{\cdot^{\cdot}} \\ & & & \alpha_{-2} & \alpha_{-1} & \alpha_0 & & \\ & & \alpha_{-2} & \alpha_{-1} & \alpha_0 & \alpha_1 & & \\ \hline & & \alpha_{-1} & \alpha_0 & \alpha_1 & \alpha_2 & & \\ & & \alpha_0 & \alpha_1 & \alpha_2 & & & \\ \cdot^{\cdot^{\cdot}} & & & & \vdots & & & \end{array}\right].$$

If $\alpha_k \geqslant 0$ for each integer k, then $\|T\| = \|H\| = \Sigma_{-\infty}^{\infty}\alpha_k$.

Proof. Let S_k be the matrix in the same form as T with ones along the kth diagonal and zeros elsewhere. Then it is obvious that $\|S_k\| = 1$ and $\|T\| = \|\Sigma\alpha_k S_k\| \leqslant \Sigma\alpha_k$. On the other hand, consider the $n \times n$ Toeplitz matrix

$$X = \begin{bmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ \alpha_{-1} & \alpha_0 & \alpha_1 & & \\ \alpha_{-2} & \alpha_{-1} & \alpha_0 & & \\ \vdots & & & \ddots & \\ \alpha_{-n} & & & & \alpha_0 \end{bmatrix}$$

(as a submatrix of T) and the constant column vector $v = (1,1,\ldots,1) \in \mathbb{C}^n$. Then

$$\|T\| \geqslant (Xv, v)/\|v\|^2$$
$$= \alpha_0 + \left(1 - \frac{1}{n}\right)(\alpha_1 + \alpha_{-1}) + \left(1 - \frac{2}{n}\right)(\alpha_2 + \alpha_{-2}) + \cdots + \frac{1}{n}(\alpha_n + \alpha_{-n}).$$

When n approaches to infinity, the right-hand side approaches to $\Sigma\alpha_k$. Thus $\|T\| \geqslant \Sigma\alpha_k$. Therefore the equality $\|T\| = \Sigma\alpha_k$ is verified.

The proof above can be modified so as to yield the similar result for H. Alternatively, let U be the unitary matrix in the same form as H with ones along the main cross-diagonal and zeros everywhere; then it follows that $T = HU$, and thus $\|T\| = \|H\|$ as desired.

Solution. (1) Euler's formula will be used twice here. First by direct computation, $(A^{\#})^2$ is in the same form as T in the Lemma with $\alpha_0 = \pi^2/3$ and $\alpha_k = 2/k^2$ if k is a nonzero integer. Since $A^{\#}$ is hermitian, it follows that

$$\|A^{\#}\|^2 = \|(A^{\#})^2\| = \sum_{-\infty}^{\infty} \alpha_k = \pi^2.$$

Therefore $\|A^{\#}\| = \pi$ as desired.

(2) follows from the Lemma.

Note. The fact $\|B\| < \pi$ and $\|B_0\| = \infty$ is well known (see [**4**, § 9.6]).

Acknowledgements. This research was supported in part by NSERC of Canada. The author would like to express his thanks to Professors Paul Halmos, John Todd, and Olga Taussky Todd for stimulating communications on the Hilbert matrix.

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A GROUP WITH CONSTANT GROWTH RATE

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1. Introduction. The theory of direct products in groups and related constructions in algebra is a rich and varied field. A sample list of some works (by no means complete) is included in the references. Problems often center on obtaining certain invariants for the decompositions involved. Sometimes the problems that arise are easy to state and comprehend. However, as in elementary number theory, some of these problems await fresh insights and new methods for their solutions. Sometimes the results obtained are somewhat anomalous insofar as they show that direct products do not always behave as one might expect. It is only after looking at this construction for some time that one begins to accept a general principle. This is: Almost anything can happen! In this paper we will discuss a result which seems to violate the intuition of our formative years.

2. A Review of Some Basics. Before stating the exact problem to be discussed, a brief review of the fundamental concepts which occur might be useful. If A and B are isomorphic groups, we write $A \approx B$.

2.1 *Direct Products*. We say that a group G is the direct products of its n subgroups

$$G_1, G_2, \ldots, G_n$$

if three conditions hold. In the first place, every element g in G must be expressible as a product

$$g = g_1 \cdot g_2 \cdots g_n, \quad g_i \in G_i. \tag{1}$$

Secondly, if $w \in G_i$ and $v \in G_j$ and $i \neq j$ then $wv = vw$. Finally the expression in (1) must be unique. That is, if in addition to (1) we also have

$$g = \bar{g}_1 \cdot \bar{g}_2 \cdots \bar{g}_n, \quad \bar{g}_i \in G_i$$

then $\bar{g}_i = g_i$ for all i. When these three conditions hold we write

$$G = G_1 \times G_2 \times \cdots \times G_n.$$

The above definition (internal definition) starts with the group G. A natural question which arises is the following. If one is given n arbitrary groups

$$G_1, G_2, \ldots, G_n \tag{2}$$

which are not necessarily distinct, is there a group G with

$$\bar{G} = \bar{G}_1 \times \bar{G}_2 \times \cdots \times \bar{G}_n$$

with $\bar{G}_i \approx G_i$ for all i? The answer is yes. Define $\bar{G}$ as the cartesian product of the G_i. That is, $\bar{G}$ is the set of all n-tuples α of the form

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