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Solutions of the Damped Oscillator Fokker–Planck Equation

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***Abstract*—The quantum theory of damping is presented and illustrated by means of a driven damped harmonic oscillator. The theory is formulated in the coherent state representation which illustrates very vividly the nearly classical nature of the problem. In this representation the reduced system density operator equation becomes a Fokker–Planck equation. Green's function solutions are found for the damped oscillator in closed form and as an eigenfunction expansion. In addition, a quantum regression theorem due to Lax is derived in the coherent state representation. The theorem allows two-time averages to be computed from one-time averages.**

I. Introduction

THE QUANTUM THEORY of *noisy* radiation fields has recently received considerable impetus from studies of the statistical properties of laser radiation.[1]–[3] A host of schemes have been proposed and employed, several with success, to compute these properties. In this paper, the authors wish to outline one such scheme in a self-contained presentation and apply it to a simple physical system. Although much of the material presented here has been given in work dealing with the gas laser,[3]–[6] we feel that the physical complexity of realistic laser models obscures the essential logic and simplicity of the method of attack. Thus, in the following sections the unencumbered theory of boson systems interacting weakly with large reservoirs in thermal equilibrium will be presented. As a concrete, but physically simple example of the manner in which the method leads to results, the statistical properties of a damped driven harmonic oscillator are investigated.

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In addition to clarifying the theory, some new exact expressions for the reduced density operator in the P representation[7] corresponding to the damped driven oscillator are found. A new proof of a regression theorem proved first by Lax[8] is given, and its relation to the existence of a reduced conditional probability and to the master equation for the reduced density operator is shown.

Sections II and III form a review of the coherent state representation,[7],[9],[10] and various ordering techniques[11] useful for computations involving coherent states. In Section IV, we derive a master equation for the reduced density operator by an approach similar to that described by Abragam,[12] but more general. Section V includes a description of the model which is used as an example throughout the paper. The master equation is cast into the P representation in Section VI using the techniques of Appendix A. The resulting form is that of a classical Fokker–Planck equation. In Section VII, exact solutions of the equation for the model are found, and in Section VIII the corresponding characteristic function and the first few moments of the corresponding distribution function are also found. Section IX extends our results to the case of two-time averages, and contains a discussion of Lax's regression theorem, and a computation of two-time amplitude and intensity correlations for the model.

II. The Coherent State Representation[7],[9],[13],[14]

A single mode of a radiation field of frequency ω_c in a cavity is described[15] by a Hamiltonian $H = \hbar\omega_c a^\dagger a$ where a and $a^\dagger$ satisfy the commutation relation $[a, a^\dagger] = 1$. The photon number or n representation is made up of the complete orthonormal set of states $|n\rangle$ which satisfy the eigenvalue equation $a^\dagger a\,|n\rangle = n\,|n\rangle$, where $n = 0, 1, 2 \cdots$. The coherent or minimum uncertainty state representation has as basis vectors the eigenstates of a with complex eigenvalues α

$$a\,|\alpha\rangle = \alpha\,|\alpha\rangle$$
$$\langle\alpha|\,a^\dagger = \alpha^*\,\langle\alpha|. \tag{1}$$

These states may be expanded in the n representation[7],[9] as

$$|\alpha\rangle = e^{-1/2\alpha^*\alpha}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\,|n\rangle = e^{-1/2\alpha^*\alpha}\sum_{n=0}^{\infty}\frac{(\alpha a^\dagger)^n}{n!}\,|0\rangle$$
$$= e^{-1/2\alpha^*\alpha}e^{\alpha a^\dagger}\,|0\rangle. \tag{2}$$

The factor $e^{-1/2\alpha^*\alpha}$ insures normalization

$$\langle\alpha\mid\alpha\rangle = 1. \tag{3}$$

To pass from one representation to the other, one uses the transformation function given by (2)

$$\langle n\mid\alpha\rangle = \frac{\alpha^n}{\sqrt{n!}}\exp\left(-\tfrac{1}{2}\alpha^*\alpha\right) \tag{4}$$

and the completeness relations

$$\iint\frac{d^2\alpha}{\pi}\,|\alpha\rangle\langle\alpha| = \Sigma_n\,|n\rangle\langle n| = 1 \tag{5}$$

where $d^2\alpha \equiv d\,\mathrm{Re}\,(\alpha)\cdot d\,\mathrm{Im}\,(d)$, and 1 is the unit operator. This completeness relation for the $|\alpha\rangle$ states may be verified explicitly using the expansion (2). Notice that the coherent states are not orthogonal

$$\langle\alpha\mid\beta\rangle = e^{-1/2(|\alpha|^2+|\beta|^2)+\beta\alpha^*} \neq \delta(\alpha-\beta)$$

but this does not impair their usefulness. The projection operator

$$\Lambda \equiv |\alpha\rangle\langle\alpha| = e^{-\alpha^*\alpha}e^{\alpha a^\dagger}\,|\alpha\rangle\langle\alpha|\,e^{\alpha^* a} \tag{6}$$

is used frequently. Some important relations involving Λ are derived in Appendix A.

The coherent states used in this paper are examples of a general class of *overcomplete* states studied extensively by Glauber, Klauder, and Sudarshan.[7],[13],[14] We refer the interested reader to this work for further detail.

III. Ordered Operator Techniques[11],[16]

The completeness relation (5) leads to an integral relation for the trace of an operator in the coherent state representation

$$\mathrm{Tr}\,(M) = \mathrm{Tr}\,M\int\frac{d^2\alpha}{\pi}\,|\alpha\rangle\langle\alpha| = \int\frac{d^2\alpha}{\pi}\,\langle\alpha|\,M\,|\alpha\rangle \tag{7}$$

where we have used $\mathrm{Tr}\,|u\rangle\langle v| = \langle v\mid u\rangle$. Since evaluation of traces is important in quantum statistical mechanics, (7) provides motivation for studying the matrix elements of operators in the $|\alpha\rangle$ representation. This study leads naturally to the concept of ordered operators.

Consider an operator function $M(a^\dagger, a)$ of the operators $a^\dagger$ and a which can be expanded in a power series about $a^\dagger = a = 0$. Using the commutation relations $[a, a^\dagger] = 1$ we may rearrange each term in this series so that all $a^\dagger$'s precede all a's. Thus rearranged, the operator M is said to be in normal form

$$M(a^\dagger, a) = \Sigma_{rs}M_{rs}^{(n)}a^{\dagger r}a^s \equiv M^{(n)}(a^\dagger, a). \tag{8}$$

Using the property (1) of the coherent states we see from (8) that

$$\langle\alpha|\,M(a^\dagger, a)\,|\alpha\rangle = \bar{M}^{(n)}(\alpha^*, \alpha). \tag{9}$$

Here, the bar over $\bar{M}^{(n)}$ indicates that M is no longer an operator. This computation shows that the problem of finding matrix elements in the coherent state representation is equivalent to that of finding the normal form of an operator function. In fact, the coefficients M_{rs} in (8) uniquely define the function

$$\bar{M}^{(n)}(\alpha^*, \alpha) = \Sigma_{rs}M_{rs}^{(n)}\alpha^{*r}\alpha^s.$$

To pass in the opposite direction from $\bar{M}^{(n)}(\alpha^*, \alpha)$ to $M(a^\dagger, a)$, it is useful to define a *normal ordering* operator $\mathfrak{N}$ with the property

$$\mathfrak{N}[\bar{M}^{(n)}(\alpha^*, \alpha)] = M^{(n)}(a^\dagger, a) = M(a^\dagger, a). \tag{10}$$

This operator replaces α^* and α by $a^\dagger$ and a in the power series expansion of $\bar{M}^{(n)}(\alpha^*, \alpha)$ after all α's have been placed to the right of all α^*'s. We may think of $\bar{M}^{(n)}(\alpha^*, \alpha)$ as describing a *classical* system which is equivalent to the quantum system described by $M(a^\dagger, a)$. For example, the vector potential describing one mode of the radiation field is

$$\mathbf{A}(\mathbf{r}, t) = [ae^{i(\mathbf{k}\cdot\mathbf{r}-\omega_c t)} + a^\dagger e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_c t)}]$$

where a and $a^\dagger$ are operators. The diagonal matrix element in the α representation is

$$\langle\alpha|\,\mathbf{A}(\mathbf{r}, t)\,|\alpha\rangle = [\alpha e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_c t)} + \alpha^* e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_c t)}]$$

which is exactly the same form as the classical vector potential where α and α^* are the complex calssical amplitudes of the field mode. In contrast, using n representation we have

$$\langle n|\,\mathbf{A}(\mathbf{r}, t)\,|n\rangle = [\sqrt{n}\,e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_c t)}\langle n\mid n-1\rangle$$
$$+ \sqrt{n+1}\,e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_c t)}\langle n\mid n+1\rangle].$$

The classical nature at high field strengths is now obscured. When many photons are involved, one gains greater physical insight by using the α representation; whereas, for only a few photons (less than three or four), it is better to use the n representation.

Another useful formula for the trace is based upon a different operator ordering. If all a's are commuted to the *left* of all $a^\dagger$'s in the power series expansion of $M(a^\dagger, a)$, we find M in *antinormal* form

$$M(a^\dagger, a) = \Sigma_{rs} M_{rs}^{(a)} a^r a^{\dagger s} = M^{(a)}(a^\dagger, a). \tag{11}$$

This may be rewritten using the completeness relation as

$$\begin{aligned} M(a^\dagger, a) &= \Sigma_{rs} M_{rs}^{(a)} \int \frac{d^2\alpha}{\pi} a^r |\alpha\rangle\langle\alpha| a^{\dagger s} \\ &= \int \frac{d^2\alpha}{\pi} \bar{M}^{(a)}(\alpha^*, \alpha) |\alpha\rangle\langle\alpha| \end{aligned} \tag{12}$$

where

$$\bar{M}^{(a)}(\alpha^*, \alpha) = \Sigma_{rs} M_{rs}^{(a)} \alpha^r \alpha^{*s}. \tag{13}$$

Again using $\text{Tr } |u\rangle\langle v| = \langle u | v\rangle$, we find

$$\text{Tr } M(a^\dagger, a) = \int \frac{d^2\alpha}{\pi} \bar{M}^{(a)}(\alpha^*, \alpha). \tag{14}$$

This is to be compared to (7) which we can now write as

$$\text{Tr } M(a^\dagger, a) = \int \frac{d^2\alpha}{\pi} \bar{M}^{(n)}(\alpha^*, \alpha) \tag{15}$$

using (9). Thus, the trace of M may be computed as an integral if either its normal or its antinormal form is known. An operator $\mathcal{A}$ analogous to $\mathfrak{N}$ may be defined through an equation similar to (10)

$$\mathcal{A}[\bar{M}^{(a)}(\alpha^*, \alpha)] = M^{(a)}(a^\dagger, a) = M(a^\dagger, a). \tag{16}$$

If two operators $M(a^\dagger, a)$ and $N(a^\dagger, a)$ are cast into opposite orders, the following result for the trace of their product is obtained

$$\begin{aligned} \text{Tr } M(a^\dagger, a)N(a^\dagger, a) &= \Sigma_{rstu} M_{rs}^{(a)} N_{rs}^{(n)} \text{ Tr } (a^r a^{\dagger s} a^{\dagger t} a^u) \\ &= \int \frac{d^2\alpha}{\pi} \Sigma_{rstu} M_{rs}^{(a)} N_{rs}^{(n)} \alpha^{r+u} \alpha^{*s+t} \\ &= \int \frac{d^2\alpha}{\pi} \bar{M}^{(a)}(\alpha^*, \alpha) \bar{N}^{(n)}(\alpha^*, \alpha) \end{aligned} \tag{17}$$

where $\text{Tr } AB = \text{Tr } BA$ are used. Casting the a's and $a^\dagger$'s in the opposite order we find in the same way

$$\text{Tr } M(a^\dagger, a)N(a^\dagger, a) = \int \frac{d^2\alpha}{\pi} \bar{M}^{(n)}(\alpha^*, \alpha) \bar{N}^{(a)}(\alpha^*, \alpha). \tag{18}$$

The techniques developed in this section will be applied in subsequent sections to the solution of the density operator equation for damped systems. We remark that the function $\bar{M}^{(n)}(\alpha^*, \beta)$ is equal (up to a factor) to Glauber's R representation[7] of $M(a^\dagger, a)$, and that $\bar{M}^{(a)}(\alpha^*, \alpha)$ is equal to his P representation of M.

IV. Master Equation for Reduced Density Operator

In order to treat attenuation quantum mechanically, consider a system whose Hamiltonian in the absence of damping is H to be coupled weakly to a reservoir consisting of a very large number of lossless systems in thermal equilibrium at a temperature T. The reservoir is described by a Hamiltonian R, and the coupling between the system and reservoir by V. The total Hamiltonian is, therefore,

$$H_T = H + R + V. \tag{19}$$

The statistical properties of the system and reservoir are described by a density operator[17] ρ which satisfies the equation of motion

$$i\hbar \frac{\partial \rho}{\partial t} = [H + R + V, \rho]. \tag{20}$$

In general, however, we are only interested in the statistical properties of functions of the *system* operators, $M(t)$. The mean value of such an operator is given by

$$\langle M(t)\rangle \equiv \text{Tr}_{R,S}\, M\rho(t) = \text{Tr}_S\, [M\, \text{Tr}_R\, \rho(t)] \tag{21}$$

where we trace over both the system and reservoir variables. This equation shows that only the reduced density operator defined by

$$S(t) = \text{Tr}_S\, \rho(t) \tag{22}$$

is needed to compute $\langle M(t)\rangle$. We therefore would like to remove the unneeded information from (20) to obtain an equation for $S(t)$. To find this, it is convenient to transform (20) to the interaction picture by means of the transformation

$$\rho(t) = e^{-(i/\hbar)(H+R)t} \chi(t) e^{(i/\hbar)(H+R)t}. \tag{23}$$

Note that H and R commute. Then (20) yields

$$i\hbar \frac{\partial \chi}{\partial t} = [V(t), \chi] \tag{24}$$

where

$$V(t) = e^{(i/\hbar)(H+R)t} V e^{-(i/\hbar)(H+R)t} \tag{25}$$

and χ is the full density operator for system and reservoir in the interaction picture. Tracing both sides of (23) over the reservoir only, and using $\text{Tr } AB = \text{Tr } BA$, we have by (22)

$$\text{Tr}_R\, \rho = S(t) = e^{-(i/\hbar)Ht} s(t) e^{(i/\hbar)Ht} \tag{26}$$

where

$$s(t) = \text{Tr}_R\, \chi(t) \tag{27}$$

is the *reduced* density operator in the interaction picture.

Integrating both sides of (24) from $t = -\infty$ when the interaction was turned on, we obtain

$$\chi(t) = \chi(-\infty) + \frac{1}{i\hbar} \int_{-\infty}^{t} [V(t'), \chi(t')]\, dt' \tag{28}$$

so that

$$\chi(t') = \chi(-\infty) + \frac{1}{i\hbar} \int_{-\infty}^{t'} [V(t''), \chi(t'')]\, dt''. \tag{29}$$

Inserting this in the integrand in (28), we obtain

$$\begin{aligned} \chi(t) = \chi(-\infty) &+ \frac{1}{i\hbar} \int_{-\infty}^{t'} [V(t'), \chi(-\infty)]\, dt' + \left(\frac{1}{i\hbar}\right)^2 \\ &\cdot \int_{-\infty}^{t} dt' \int_{-\infty}^{t'} dt'' [V(t'), [V(t''), \chi(t'')]]. \end{aligned} \tag{30}$$

If we proceeded to iterate again in this fashion, we would obtain $\chi(t)$ in a power series in V which is just perturbation theory. Unforturnately, perturbation theory is incapable of yielding exponential decay, and we must resort to another approximation scheme. Differentiating both sides of (30) with respect to t, we find

$$\frac{\partial \chi}{\partial t} = \frac{1}{i\hbar}[V(t'), \chi(-\infty)] + \left(\frac{1}{i\hbar}\right)^2 \int_{-\infty}^{t} [V(t), [V(t''), \chi(t'')]]\, dt'' \tag{31}$$

whose trace over reservoir variables, using (27), is

$$\frac{\partial s}{\partial t} = \left(\frac{1}{i\hbar}\right)^2 \int_{-\infty}^{t} \mathrm{Tr}_R\, [V(t), [V(t'), \chi(t')]]\, dt'. \tag{32}$$

Here, we have used

$$\mathrm{Tr}_R\, [V(t'), \chi(-\infty)] = 0 \tag{33}$$

which follows from the factorization of χ at $t = -\infty$, when the interaction V is assumed *turned off*

$$\chi(-\infty) = s(-\infty) f_0(R)$$

and $f_0(R)$ is a Boltzmann distribution

$$f_0(R) = \frac{e^{-\beta R}}{\mathrm{Tr}_R\, e^{-\beta R}}$$

where $\beta = 1/kT$. In (33), $f_0(R)$ is diagonal but we assume $V(t)$ has no diagonal elements in the R representation so that the trace vanishes. Aside from this assumption which is not a real restriction, (32) is exact.

To continue, we assume that the damped system is *Markoffian* which means that its future behavior is determined only by the present and not by the past. Physically this means that exposure to the reservoir blanks out the past memory of the system. Mathematically we simply replace $\chi(t')$ on the right of (32) by $\chi(t)$. In addition, we assume that $\chi(t)$ has the form

$$\chi(t) = s(t) f_0(R) + \Delta\chi \tag{34}$$

where $\Delta\chi$ is at most of order V since if $V = 0$, $\chi(t) = s(t) f_0(R)$. Our second approximation will now be in ignoring all quantities of order higher than second in V. With these assumptions (32) reduces to

$$\frac{\partial s}{\partial t} = \left(\frac{1}{i\hbar}\right)^2 \int_{-\infty}^{t} \mathrm{Tr}_R\, [V(t), [V(t'), s(t) f_0(R)]]\, dt' \tag{35}$$

which is our *master equation* for s.[12]

If, in addition to the reservoir, there is another interaction, U which depends only on the system operators, we assume that the reduced density operator in the interaction picture satisfies

$$\frac{\partial s}{\partial t} = \left(\frac{1}{i\hbar}\right)[U(t), s] + \left(\frac{1}{i\hbar}\right)^2 \int_{-\infty}^{t} \mathrm{Tr}_R\, [V(t), [V(t'), s(t) f_0(R)\, dt']] \tag{36}$$

where

$$U(t) = e^{(i/\hbar)Ht} U e^{-(i/\hbar)Ht}. \tag{37}$$

That is, $U(t)$ is treated as though it were not influenced directly by the reservoir.

The generalization to many coupled systems, each with its own reservoir, is obvious.

V. Master Equation for the Damped Driven Oscillator

A single mode of the radiation field of frequency ω_c coupled weakly to a reservoir of oscillators at temperature T, and driven by a classical source has the Hamiltonian

$$\begin{aligned} H_T &= H + R + V + U \\ H &= \hbar\omega_c a^\dagger a \\ R &= \Sigma \hbar\omega_j b_j^\dagger b_j \\ V &= \Sigma \hbar(\kappa_j b_j a^\dagger + \kappa_j^* b_j^\dagger a) \\ U &= \hbar[p(t) a^\dagger + p^*(t) a]. \end{aligned} \tag{38}$$

The reduced density operator by (37) satisfies

$$\frac{\partial s}{\partial t} = \frac{1}{i\hbar}[U(t), s] + \left(\frac{1}{i\hbar}\right)^2 \int_{-\infty}^{t} \mathrm{Tr}_R\, [V(t), [V(t'), s(t) f_0(R)]]\, dt' \tag{39}$$

where[18]

$$U(t) = \hbar[p(t) e^{i\omega_c t} a^\dagger + p^*(t) e^{-i\omega_c t} a] \tag{40}$$

$$V(t) = \Sigma \hbar[\kappa_j b_j a^\dagger e^{i(\omega_c - \omega_j)t} + \kappa_j^* b_j^\dagger a e^{-i(\omega_c - \omega_j)t} \equiv \hbar[G(t) a^\dagger + G^\dagger(t) a]. \tag{41}$$

We shall work out in Appendix B only one of the terms in (39). The others may be obtained similarly. The result is

$$\frac{\partial s}{\partial t} = -ip(t) e^{i\omega_c t}[a^\dagger, s] - ip^*(t) e^{-i\omega_c t}[a, s] + \frac{\gamma}{2}[2asa^\dagger - a^\dagger as - sa^\dagger a] + \gamma\bar{n}[a^\dagger sa + asa^\dagger - a^\dagger as - saa^\dagger]. \tag{42}$$

The first two terms are pumping terms. The $\gamma/2$ term is a damping term which represents loss from the system; γ is defined in Appendix B. The $\gamma\bar{n}$ term is a diffusion term and represents the diffusion of thermal noise from the reservoir diffusing into the system. This identification of terms will become clear in the next section. From Appendix B we have

$$\bar{n} = \frac{1}{\exp(\hbar\omega_c/\kappa T) - 1}$$

which is the mean number of thermal photons (or phonons) in the reservoir. As $T \to 0$, $\bar{n}$ vanishes and there is no diffusion.

There are many techniques[19] available to solve equations such as (42), which result from quadratic Hamiltonians. In the following sections, we use the harmonic oscillator model described above to illustrate still another technique which is useful for more general problems. Louisell, working with Lax,[4] has applied this method with success to the problem of noise in gas lasers.

VI. Fokker–Planck Equation for $\bar{s}^{(a)}$ (α^*, α, t)

Glauber[7] has emphasized the convenience of working with the density operator in the P representation. In particular, (18) gives for the mean of a system operator $M(a^\dagger, a)$

$$\langle M \rangle = \mathrm{Tr}\, sM = \int \frac{d^2\alpha}{\pi} s^{(a)}(\alpha^*, \alpha) M^{(n)}(\alpha^*, \alpha).$$

To find an equation for $s^{(a)}(\alpha^*, \alpha)$ it is only necessary to use (12) on both sides of the master equation (35) or (36), and then apply the results of Appendix A. The procedure is best understood by working out a specific case. We do this for the damped driven oscillator (42).

From the relation

$$s(a^\dagger, a, t) = \int \bar{s}^{(a)}(\alpha^*, \alpha, t) \Lambda \frac{d^2\alpha}{\pi} \tag{43}$$

and the normalization condition Tr $s = 1$, we find

$$\mathrm{Tr}_S\, S(a, a^\dagger, t) = \mathrm{Tr}_S\, s(a, a^\dagger, t) = \int \bar{s}^{(a)}(\alpha, \alpha^*, t) \frac{d^2\alpha}{\pi} = 1 \tag{44}$$

so that $\bar{s}^{(a)}(\alpha, \alpha^*, t)$ plays the role of a *classical* probability density. If we put (43) into (42), we have

$$\int \Lambda \frac{d^2\alpha}{\pi} \frac{\partial \bar{s}^{(a)}}{\partial t} = \int \frac{d^2\alpha}{\pi} \bar{s}^{(a)} \Big\{ -ip(t)e^{i\omega_c t}[a^\dagger, \Lambda] - ip^*(t)e^{-i\omega_c t}[a, \Lambda] + \frac{\gamma}{2}[2a\Lambda a^\dagger - a^\dagger a \Lambda - \Lambda a^\dagger a] + \gamma\bar{n}[a^\dagger \Lambda a + a\Lambda a^\dagger - a^\dagger a \Lambda - \Lambda a a^\dagger] \Big\}. \tag{45}$$

We may use the results of Appendix A to show that (45) becomes

$$\int \Lambda \frac{d^2\alpha}{\pi} \frac{\partial \bar{s}^{(a)}}{\partial t} = \int \frac{d^2\alpha}{\pi} \bar{s}^{(a)} \Big\{ -ip(t)e^{i\omega_c t} \frac{\partial \Lambda}{\partial \alpha} + ip^*(t)e^{-i\omega_c t} \frac{\partial \Lambda}{\partial \alpha^*} - \frac{\gamma}{2}\left(\alpha \frac{\partial \Lambda}{\partial \alpha} + \alpha^* \frac{\partial \Lambda}{\partial \alpha^*}\right) + \gamma\bar{n} \frac{\partial^2 \Lambda}{\partial\alpha \partial\alpha^*} \Big\}. \tag{46}$$

We next integrate all the first derivative terms on the right by parts once, and the second derivative term by parts twice. The integrated parts vanish since $\bar{s}^{(a)}$ must be integrable by (44) so that $\bar{s}^{(a)}$ must vanish as α and $\alpha^* \to \infty$. We therefore obtain for (46)

$$\int \Lambda \frac{d^2\alpha}{\pi} \frac{\partial \bar{s}^{(a)}}{\partial t} = \int \Lambda \frac{d^2\alpha}{\pi} \Big\{ +ip(t)e^{i\omega_c t} \frac{\partial \bar{s}^{(a)}}{\partial \alpha} - ip^*(t)e^{-i\omega_c t} \frac{\partial \bar{s}^{(a)}}{\partial \alpha^*} + \left[\frac{\gamma}{2} \frac{\partial}{\partial \alpha}(\alpha \bar{s}^{(a)}) + \frac{\partial}{\partial \alpha^*}(\alpha^* \bar{s}^{(a)})\right] + \gamma\bar{n} \frac{\partial^2 \bar{s}^{(a)}}{\partial\alpha\partial\alpha^*} \Big\}. \tag{47}$$

Therefore the associated antinormal reduced density function satisfies the Fokker–Planck equation

$$\frac{\partial \bar{s}^{(a)}}{\partial t} = -\frac{\partial}{\partial \alpha}\left(-\frac{\gamma}{\alpha}\alpha - ip(t)e^{i\omega_c t}\right)\bar{s}^{(a)} - \frac{\partial}{\partial \alpha^*}\left(-\frac{\gamma}{2}\alpha^* + ip^*(t)e^{-i\omega_c t}\right)\bar{s}^{(a)} + \gamma\bar{n} \frac{\partial^2 \bar{s}^{(a)}}{\partial\alpha\partial\alpha^*}. \tag{48}$$

There are many discussions of Fokker–Planck equations in the literature, and the reader should consult, for example, Lax[20] for details. We remark that the terms in parenthesis play the role of components of a *drift vector*, and $\gamma\bar{n}$ that of *diffusion constant*. Clearly a similar equation can be written down for any arbitrary system. However, *only* for systems with quadratic Hamiltonians, such as those in our example, will the Fokker–Planck equation include at most second derivatives.

VII. Solutions of the Fokker–Planck Equation for Damped Driven Oscillator

A. General Solution for Harmonic Driving Function

When the driving term is sinusoidal and at the cavity frequency

$$p(t) = p_0 e^{-i\omega_c t} \tag{49}$$

the Fokker–Planck equation (48) reduces to

$$\frac{\partial \bar{s}^{(a)}}{\partial t} = \frac{\gamma}{2}\left(\alpha + ip_0 \frac{2}{\gamma}\right)\frac{\partial \bar{s}^{(a)}}{\partial \alpha} + \frac{\gamma}{2}\left(\alpha^* - ip_0^* \frac{2}{\gamma}\right)\frac{\partial \bar{s}^{(a)}}{\partial \alpha^*} + \gamma\bar{n}\frac{\partial^2 \bar{s}^{(a)}}{\partial\alpha\partial\alpha^*} + \gamma\bar{s}^{(a)} \equiv L\bar{s}^{(a)} \tag{50}$$

which is separable in the time. With

$$\bar{s}^{(a)}(\alpha, \alpha^*, t) = e^{-\lambda t} Q(\alpha, \alpha^*) \tag{51}$$

this becomes

$$LQ(\alpha, \alpha^*) = -\lambda Q(\alpha, \alpha^*). \tag{52}$$

The operator L is not self-adjoint[21] but we may make it so by taking

$$Q(\alpha, \alpha^*) = e^{-(x^2+y^2)/2} N(x, y) \tag{53}$$

where

$$\alpha + \frac{i2}{\gamma} p_0 = \sqrt{\bar{n}}\,(x + iy), \qquad \alpha^* - \frac{2i}{\gamma} p_0^* = \sqrt{\bar{n}}\,(x - iy). \tag{54}$$

This substitution yields

$$\frac{\partial^2 N}{\partial x^2} + \frac{\partial^2 N}{\partial y^2} + [\epsilon - x^2 - y^2]N(x, y) = 0 \quad (55)$$

where

$$\epsilon = \frac{4}{\gamma}\lambda + 2. \quad (56)$$

This is nothing but the Schrödinger equation for a two-dimensional isotropic harmonic oscillator. Furthermore, since

$$\int \bar{s}^{(a)}(\alpha, \alpha^*, t)\frac{d^2\alpha}{\pi} = 1 \quad (57)$$

we see that $\bar{s}^{(a)}$ and, therefore, $N(x, y)$ must vanish at $x = \pm\infty$, which are the same boundary conditions for the Schrodinger equation. The corresponding eigenfunctions and eigenvalues are well known to be

$$\epsilon = \frac{4}{\gamma}\lambda + 2 = (2n_x + 1) + (2n_y + 1) \qquad (n_x, n_y = 0, 1, 2, \cdots) \quad (58)$$

$$N_{n_x,n_y}(x, y) = K_{n_x,n_y}e^{-1/2(x^2+y^2)}H_{n_x}(x)H_{n_y}(y) \quad (59)$$

where the $H_n(\xi)$ are the hermite polynomials, and the K_{n_x,n_y} are normalizing constants. The eigenfunctions of (50) are therefore

$$\bar{s}^{(a)}_{n_xn_y}(x, y, t) = K_{n_xn_y}e^{-\gamma/2(n_x+n_y)t}e^{-(x^2+y^2)}H_{n_x}(x)H_{n_y}(y). \quad (60)$$

The integral (57) in terms of the variables x and y becomes

$$\iint_{-\infty}^{\infty} \frac{\bar{s}^{(a)}_{n_xn_y}(x, y, t)\bar{n}}{\pi}\, dx\, dy = 1. \quad (61)$$

Since

$$\int_{-\infty}^{\infty} e^{-x^2}H_n(x)H_m(x)\, dx = \delta_{nm}2^n n!\sqrt{\pi} \quad (62)$$

and

$$H_0(x) = 1 \quad (63)$$

we have by (60) to (63)

$$\frac{\bar{n}K_{n_xn_y}}{\pi}e^{-\gamma/2(n_x+n_y)t}\int_{-\infty}^{\infty} e^{-x^2}H_{n_x}(x)\, dx \int_{-\infty}^{\infty} e^{-y^2}H_{n_y}(y)\, dy = \bar{n}K_{00}\,\delta_{n_x0}\,\delta_{n_y0} = 1. \quad (64)$$

We must therefore let

$$K_{00} = \frac{1}{\bar{n}} \quad (65)$$

so that for $n_x = n_y = 0$, (60) becomes

$$\bar{s}^{(a)}_{00}(x, y, t) = \frac{1}{\bar{n}}e^{-(x^2+y^2)}$$

for the *ground-state* eigenfunction. Since it is the only nondamped eigenfunction, this represents the steady-state solution to our Fokker–Planck equation.

From the properties of the self-adjoint equation[21] for $N_{n_xn_y}(x, y)$ [see (55)], we know that the eigenfunctions are complete

$$\sum_{m=0}^{\infty} \frac{1}{2^n n!\sqrt{\pi}}e^{-1/2(x^2+x'^2)}H_n(x)H_n(x') = \delta(x - x'). \quad (66)$$

We may use this completeness relation to obtain a solution for the Fokker–Planck equation such that at $t = 0$, $x = x'$ and $y = y'$, or $\alpha = \alpha'$ and $\alpha^* = \alpha'^*$. This solution, the conditional probability, is given by

$$\bar{s}^{(a)}(\alpha, \alpha^*, t \mid \alpha', \alpha'^*, 0) = \sum_{n_x,n_y=0}^{\infty} \frac{\bar{s}^{(a)}_{n_x,n_y}(x, y, t)\bar{s}^{(a)}_{n_x,n_y}(x', y', 0)}{\bar{n}\bar{s}^{(a)}_{00}(x, y)}. \quad (67)$$

It is easily seen from (67), (60), and (61) that at $t = 0$ the aforementioned conditional probability reduces to

$$\bar{s}^{(a)}(\alpha, \alpha^*, 0 \mid \alpha', \alpha'^*, 0) = \delta(\alpha - \alpha')\,\delta(\alpha^* - \alpha^{*\prime}) = \delta(x - x')\,\delta(y - y')$$

provided

$$K_{n_x,n_y} = \frac{1}{\sqrt{2^{n_x}n_x!\sqrt{\pi}\,2^{n_y}n_y!\sqrt{\pi}}}.$$

The solution for an arbitrary initial distribution $\bar{s}^{(a)}(\alpha', \alpha'^*, 0)$ is

$$\bar{s}^{(a)}(\alpha, \alpha^*, t) = \int \bar{s}^{(a)}(\alpha, \alpha^*, t \mid \alpha', \alpha'^*, 0)\bar{s}^{(a)}(\alpha', \alpha'^*, 0)\frac{d^2\alpha'}{\pi}. \quad (68)$$

B. General Solution for Arbitrary Driving Function

When $p(t)$ is arbitrary, the technique mentioned does not work since the equation is not separable in the time, and we resort to another technique. We would like to find a solution of (48) subject to the initial conditions $\alpha = \alpha'$ and $\alpha^* = \alpha'^*$ at $t = 0$. Note that we may represent the $\bar{p}$ function as

$$\lim_{\epsilon\to\infty} \epsilon e^{-\epsilon(\alpha-\alpha')(\alpha^*-\alpha'^*)} = \delta(\alpha - \alpha')\,\delta(\alpha^* - \alpha'^*)$$

such that

$$\int \delta(\alpha - \alpha')\,\delta(\alpha^* - \alpha'^*)\frac{d^2\alpha}{\pi} = 1.$$

We therefore are led to a trial solution of the form

$$\bar{s}^{(a)}(\alpha, \alpha^*, t \mid \alpha', \alpha'^*, 0) = e^{G(t)} \quad (69)$$

where

$$G(t) = -\frac{[\alpha - \eta(t)][\alpha^* - \eta^*(t)]}{\zeta(t)} + \ln \nu(t) \quad (70)$$

and

$$\eta(0) = \alpha'; \quad \eta^*(0) = \alpha'^*, \quad \zeta(0) = \frac{1}{\epsilon}; \quad \nu(0) = \epsilon. \quad (71)$$

When we put (69) into (48), and equate the coefficients of equal powers of α and α^*, we obtain the coupled set of equations

$$\begin{aligned}\frac{d\zeta}{dt} &= -\gamma\zeta + \gamma\bar{n} \\ \frac{d\eta}{dt} &= -\frac{\gamma}{2}\eta - ip(t)e^{-i\omega_c t} \\ \frac{1}{\nu}\frac{d\nu}{dt} &= -\frac{1}{\zeta}\frac{d\zeta}{dt}\end{aligned} \tag{72}$$

after minor algebraic simplifications. The solutions are

$$\begin{aligned}\zeta(t) &= \frac{1}{\epsilon} + \bar{n}(1 - e^{-\gamma t}) \underset{\epsilon\to\infty}{\longrightarrow} \bar{n}(1 - e^{-\gamma t}) \\ \eta(t) &= \alpha' e^{-(\gamma/2)t} + w(t)e^{i\omega_c t} \\ \nu(t) &= \frac{1}{\zeta(t)} \underset{\epsilon\to\infty}{\longrightarrow} \frac{1}{\bar{n}(1 - e^{-\gamma t})} = \frac{1}{\zeta(t)}\end{aligned} \tag{73}$$

where

$$w(t) = -i\int_0^t p(t - t')e^{-(i\omega_c + \gamma/2)t'}\,dt'. \tag{74}$$

Accordingly, the conditional probability distribution function is

$$\bar{s}^{(a)}(\alpha, \alpha^* t \mid \alpha', \alpha'^*, 0) = \frac{1}{\bar{n}(1 - e^{-\gamma t})}\exp -\frac{|\alpha - \alpha' e^{-(\gamma/2)t} - w(t)e^{i\omega_c t}|^2}{\bar{n}(1 - e^{-\gamma t})}. \tag{75}$$

Clearly, as $t \to 0$, $w(t) \to 0$, and we have a δ function as required, we may use this in (68) to obtain the solution for an arbitrary initial distribution. In the special case of a sinusoidal driving term given by (49), $w(t)$ becomes

$$e^{i\omega_c t}w(t) = -\frac{2i}{\gamma}p_0(1 - e^{-(\gamma/2)t}) \tag{76}$$

in which case (67) and (75) become identical.

VIII. The Characteristic Function[22]

The characteristic function is defined by

$$\begin{aligned}C^{(a)}(\xi\eta, \xi\eta^*, t) &= \mathrm{Tr}_{R,S}\ \rho(t)e^{i\xi\eta^* a^\dagger}e^{i\xi\eta a} \\ &= \mathrm{Tr}_S\ S(a, a^\dagger, t)e^{i\xi\eta^* a^\dagger}e^{i\xi\eta a}\end{aligned} \tag{77}$$

where we have used (22). From the characteristic function we may calculate all moments of the form

$$\langle a^{\dagger l}(t)a^m(t)\rangle = \frac{\partial^{l+m}}{\partial(i\xi\eta^*)^l\,\partial(i\xi\eta)^m}C(\xi\eta, \xi\eta^*, t)|_{\xi=0}. \tag{78}$$

If we use (17), we see that

$$C^{(a)}(\xi\eta, \xi\eta^*, t) = \int \frac{d^2\alpha}{\pi}\bar{S}^{(a)}(\alpha, \alpha^*, t)e^{i\xi(\eta\alpha + \eta^*\alpha^*)} \tag{79}$$

so that $C^{(a)}$ and $\bar{S}^{(a)}$ are Fourier transform pairs. Accordingly, we easily see that

$$\bar{S}^{(a)}(\alpha, \alpha^*, t) = \int e^{-i\xi(\eta\alpha + \eta^*\alpha^*)}C^{(a)}(\xi\eta, \xi\eta^*, t)\xi^2\frac{d^2\eta}{\pi} \tag{80}$$

so that if we know the characteristic function, we may obtain the associated function for the reduced density operator,[23] and

$$S^{(a)}(a, a^\dagger, t) = \mathcal{A}\{\bar{S}^{(a)}(\alpha, \alpha^*, t)\}. \tag{81}$$

Thus, the characteristic function uniquely determines the density operator and vice versa.

We have obtained the reduced density operator in the interaction picture. In the interaction picture (77) becomes

$$\begin{aligned}&C^{(a)}(\xi\eta, \xi\eta^*, t) \\ &= \mathrm{Tr}_S\ s(a, a^\dagger, t)\exp(i\xi\eta^* a^\dagger e^{i\omega_c t})\exp(i\xi\eta a e^{-i\omega_c t}) \\ &= \int \bar{s}^{(a)}(\alpha, \alpha^*, t)\exp i\xi(\eta^*\alpha^* e^{i\omega_c t} + \eta\alpha e^{-i\omega_c t})\frac{d^2\alpha}{\pi}.\end{aligned} \tag{82}$$

We may use (68) and (75) to obtain $C^{(a)}(\xi\eta, \xi\eta^*, t)$. We may easily obtain the following

$$\begin{aligned}\langle a(t)\rangle &= e^{-(i\omega_c + \gamma/2)t}\langle a(0)\rangle + w(t) \\ \langle a^\dagger(t)\rangle &= e^{+(i\omega_c - \gamma/2)t}\langle a^\dagger(0)\rangle + w^*(t) \\ \langle a^\dagger(t)a(t)\rangle &= \bar{n}(1 - e^{-\gamma t}) + e^{-\gamma t}\langle a^\dagger(0)a(0)\rangle + |w(t)|^2 \\ &\quad + w(t)e^{(i\omega_c - \gamma/2)t}\langle a^\dagger(0)\rangle + w^*(t)e^{-(i\omega_c + \gamma/2)t}\langle a(0)\rangle.\end{aligned} \tag{83}$$

These give the mean value of the field and the mean number of quanta in the field at time t. In the steady state ($\gamma t \gg 1$), these means reduce to

$$\begin{aligned}\langle a(t)\rangle &= w(t) \\ \langle a^\dagger(t)\rangle &= w^*(t) \\ \langle a^\dagger(t)a(t)\rangle &= \bar{n} + |w(t)|^2.\end{aligned} \tag{84}$$

All previous information in the field is lost, leaving only the driving terms, whereas the mean number of field quanta comes into equilibrium with the reservoir plus a driving term contribution.

IX. Two-Time Averages

The power spectrum and the amplitude spectrum are derived from the Fourier transforms of the two-time correlation functions $\langle b^\dagger(t)b(0)\rangle$ and $\langle b^\dagger(0)b^\dagger(t)b(t)b(0)\rangle$, respectively. To obtain these two-time averages we must invoke a theorem due to Lax[8] which says that under certain conditions (Markoffian behavior is sufficient), one- and two-time averages obey the same equation of motion. In Part A of this section we show that this result is *always* true, but that only under the conditions stated by Lax are the equations of motion the macroscopic ones. Part B includes computation of the amplitude and intensity correlation functions.

A. An Exact Regression Theorem

The time development of the exact operator ρ is given by (20) which has the formal solution

$$\rho(t) = e^{-(i/\hbar)H_T t}\rho(0)e^{(i/\hbar)H_T t} \tag{85}$$

when H_T is time independent in the Schrodinger picture. If H_T includes time dependent driving terms, $\rho(t)$ may still be written as

$$\rho(t) = U(t, 0)\rho(0)U(0, t) = U(t, t')\rho(t')U(t', t). \quad (86)$$

Here, $U(t, t')$ satisfies

$$\begin{aligned} i\hbar \frac{\partial U}{\partial t} &= H_T U \\ U(t, t) &= 1 \\ U(t, t') &= U^{-1}(t', t) \\ U(t, t'') &= U(t, t')U(t', t'') \end{aligned} \quad (87)$$

and reduces to $e^{-(i/\hbar)H_T t}$ for H_T time independent. In the following discussion, a stands for the set of operators $\{a_i\}$ required to describe the system in question.

Equation (12) may be used to rewrite (86) as

$$\rho_t(a^\dagger, a) = \int \frac{d^2\alpha}{\pi}\, U_{tt'}(a^\dagger, a)\bar{p}_{t'}^{(a)}(\alpha^*, \alpha)\ |\alpha\rangle\langle\alpha|\ U_{t't}(a^\dagger, a)$$

where we have changed notation in an obvious way for compactness. This equation has the following diagonal matrix element in the coherent state representation

$$\bar{p}_t^{(n)}(\beta^*, \beta) = \int \frac{d^2\alpha}{\pi}\, \bar{U}_{tt'}^{(n)}(\beta^*, \alpha)\bar{U}_{tt'}^{(n)}(\alpha^*, \beta)\bar{p}_{t'}^{(a)}(\alpha^*, \alpha).$$

Suppose we convert both sides of this equation to an operator function of b^+ and b only using (10). Then if

$$\mathcal{V}_{tt'}(b^+, b, \alpha^*, \alpha) = \mathfrak{N}_b[\bar{U}_{tt'}^{(n)}(\beta^*, \alpha)\bar{U}_{t't}^{(n)}(\alpha^*, \beta)]$$

we have

$$\rho_t(b^+, b) = \int \mathcal{V}_{tt'}(b^+, b, \alpha^*, \alpha)\bar{p}_{t'}^{(a)}(\alpha^*, \alpha).$$

Then the P representative of this equation, obtained from the antinormal form of $\mathcal{V}$ as in (11) and (13), plays the role of a conditional probability for $\bar{p}_t^{(a)}$

$$\bar{p}_t^{(a)}(\beta^*, \beta) = \int \frac{d^2\alpha}{\pi}\, \bar{\mathcal{V}}_{tt'}^{(a)}(\beta^*, \beta \mid \alpha^*, \alpha)\bar{p}_{t'}^{(a)}(\alpha^*, \alpha). \quad (88)$$

The two-time average $\langle M(t)N(t')\rangle$ of two operators in the Heisenberg picture may be written as

$$\begin{aligned} M(t)N(t') &= \mathrm{Tr}\ \rho_0 M_t N_{t'} \\ &= \mathrm{Tr}\ \rho_0 U_{0t} M_0 U_{tt'} N_0 U_{t'0} \\ &= \mathrm{Tr}\ U_{tt'} N_0 \rho_{t'} U_{t't} M_0 \end{aligned} \quad (89)$$

where we have used the cyclic property of the trace, (86) and (87), and

$$M_t = U_{0t} M_0 U_{t0}.$$

The first four factors in (89) may be treated exactly as (86) was above, with $\rho_{t'}$ replaced by $N_0\rho_{t'}$. These factors have the P representative

$$\begin{aligned} &(\overline{U_{tt'}N_0\rho_{t'}U_{t't}M_0})^{(a)}(\beta^*, \beta) \\ &\quad = \int \frac{d^2\alpha}{\pi}\, \bar{\mathcal{V}}_{tt'}^{(a)}(\beta^*, \beta \mid \alpha^*, \alpha)(\overline{N_0\rho_{t'}})^{(a)}(\alpha^*, \alpha). \end{aligned}$$

Therefore, using (17) for the trace of a product, we find

$$\langle M_t N_{t'}\rangle = \int \frac{d^2\alpha}{\pi}\int \frac{d^2\beta}{\pi}\, \bar{\mathcal{V}}_{tt'}^{(a)}(\beta^*, \beta \mid \alpha^*, \alpha) \cdot(\overline{N_0\rho_{t'}})^{(a)}(\alpha^*, \alpha)\bar{M}_0^{(n)}(\beta^*, \beta). \quad (90)$$

The single-time average M_t is obtained by setting $N = 1$ in (90)

$$\langle M_t\rangle = \int \frac{d^2\alpha}{\pi}\int \frac{d^2\beta}{\pi}\, \bar{\mathcal{V}}_{tt'}^{(a)} \cdot(\beta^*, \beta \mid \alpha^*, \alpha)\bar{p}_{t'}^{(a)}(\alpha^*, \alpha)\bar{M}_0^{(n)}(\beta^*, \beta). \quad (91)$$

That $\bar{\mathcal{V}}_{tt'}^{(a)}$ contains the essential time dependence in both cases is an expression of our exact *regression theorem* connecting one- and two-time averages.

Now suppose that of the set of operators $\{a_i\}$ represented by a in the preceding paragraph, the first N's are system operators, denoted collectively by a_s, and the remaining are reservoir operators denoted by a_R. With this notation, we trace (88) over reservoir variables in an attempt to find a conditional probability for the reduced density operator $S(t)$ defined in (22)

$$\bar{S}_t^{(a)}(\beta_s^*, \beta_s) = \int \frac{d^2\alpha_s}{\pi}\int \frac{d^2\alpha_R}{\pi} \cdot\left[\int \frac{d^2\beta_R}{\pi}\, \bar{\mathcal{V}}_{tt'}^{(a)}(\beta^*, \beta \mid \alpha^*\alpha)\right]\bar{p}_{t'}^{(a)}(\alpha^*, \alpha). \quad (92)$$

It is clearly impossible even to speak of a conditional probability in this case unless $\bar{p}_t^{(a)}$ factors as

$$\bar{p}_t^{(a)}(\alpha^*, \alpha) = \bar{R}_t^{(a)}(\alpha_R^+, \alpha_R)\bar{S}_t^{(a)}(\alpha_s^*, \alpha_s). \quad (93)$$

In this case,

$$\bar{S}_t^{(a)}(\beta_s^*, \beta_s) = \int \frac{d^2\alpha}{\pi}\, \bar{S}_{tt'}^{(a)}(\beta_s^*\beta_s \mid \alpha_s^*\alpha_s)\bar{S}_{t'}^{(a)}(\alpha_s^*, \alpha_s) \quad (94)$$

where

$$\bar{S}_{tt'}^{(a)}(\beta_s^*, \beta_s \mid \alpha_s^*\alpha_s) = \int \frac{d^2\alpha_R}{\pi}\int \frac{d^2\beta_R}{\pi} \cdot\bar{\mathcal{V}}_{tt'}^{(a)}(\beta^*, \beta \mid \alpha^*, \alpha)\bar{R}_{t'}^{(a)}(\alpha_R^*, \alpha_R) \quad (95)$$

is the Schrodinger picture conditional probability corresponding to that in the interaction picture introduced in Section VII.

In general, of course, ρ_t does *not* factor as in (93). In Section IV, however, we found that the nonfactoring part $\Delta\chi$ of the density operator played no role to second order in the weak coupling to the reservoir. Consequently, assuming ρ_t factors are consistent with our previous approximation, led to the master equations (35) and (36).

A procedure identical to that which led from (88) to (94) gives for (90)

$$\langle M_t N_{t'}\rangle = \int \frac{d^2\alpha_S}{\pi}\int \frac{d^2\beta_S}{\pi}\, \bar{S}_{tt'}(\beta_s^*, \beta_s \mid \alpha_s^*, \alpha_s) \cdot(\overline{N_0 S_{t'}})^{(a)}(\alpha_s^*, \alpha_s)\bar{M}_0^{(n)}(\beta_s^*, \beta_s) \quad (96)$$

if M and N depend only on system operators. This equation was first derived by Lax in a rather different form.

B. Two-Time Averages for the Damped Driven Oscillator

In Section VII, we found explicit expressions for the interaction picture form $\bar{s}_{tt'}$ of the conditional probability $\bar{S}_{tt'}$ employed above. Equation (96) remains valid if all operators are transformed to the interaction picture

$$\langle M_t N_{t'}\rangle = \int \frac{d^2\alpha_S}{\pi} \int \frac{d^2\beta_S}{\pi}\, \bar{s}_{tt'}(\beta_s^*, \beta_s \mid \alpha_s^*, \alpha_s) \cdot \overline{(\tilde{N}_{t'} s_{t'})}^{(a)}(\alpha_s^*, \alpha_s)\bar{\tilde{M}}_t^{(n)}(\beta_s^*\beta_s). \quad (97)$$

Here, $\tilde{N}_{t'}$ and $\tilde{M}_{t'}$ are in the interaction picture.

Using this formula and (75) for the conditional probability, we may compute the amplitude correlation function $\langle b^\dagger(t)b(0)\rangle$ without difficulty as

$$\langle b^\dagger(t)b(0)\rangle = \int \frac{d^2\alpha}{\pi} \int \frac{d^2\beta}{\pi}\, \bar{s}_{t0} \cdot (\beta^*, \beta \mid \alpha^*, \alpha)\alpha\bar{s}_0^{(a)}(\alpha^*, \alpha)\beta^* e^{i\omega_c t}.$$

The β integrals are Gaussian in this case and can be performed explicitly. Omitting details, we find

$$\langle b^\dagger(t)b(0)\rangle = e^{i\omega_c t}e^{-(\gamma/2)t}\langle b^\dagger b\rangle_0 + w^*(t)\langle b\rangle_0 \quad (98)$$

where $\langle\cdots\rangle_0$ denotes an average taken at $t = 0$, and $w(t)$ is given by (74). Notice that the steady-state correlation coefficient is proportional to $w^*(t)$.

Before we can compute the intensity correlation function $\langle b^\dagger(0)b^\dagger(t)b(t)b(0)\rangle$ we must find a formula analogous to (96) or (97) for the two-time average

$$\langle L_{t'} M_t N_{t'}\rangle = \text{Tr}\, N_{t'}\rho_0 L_{t'} M_t = \text{Tr}\, U_{tt'}(N_0\rho_{t'}L_0)U_{t't}M_0.$$

This now has the form of (89) with $N_0\rho_{t'}$ replaced by $N_0\rho_{t'}L_0$, and all the arguments given above are valid here. Thus, going immediately to the interaction picture, we have

$$\langle L_{t'} M_t N_{t'}\rangle = \int \frac{d^2\alpha_s}{\pi} \int \frac{d^2\beta_s}{\pi}\, \bar{s}_{tt'}^{(a)} \cdot (\beta_s^*\beta_s \mid \alpha_s^*\alpha_s)\overline{(\tilde{N}_{t'} s_{t'} \tilde{L}_{t'})}^{(a)}(\alpha_s^*, \alpha_s)\bar{\tilde{M}}_t^{(n)}(\beta_s^*, \beta_s). \quad (99)$$

In particular, the intensity correlation function is

$$\langle b_0^\dagger b_t^\dagger b_t b_0\rangle = \int \frac{d^2\alpha}{\pi} \int \frac{d^2\beta}{\pi}\, \bar{s}_{t0}^{(a)} \cdot (\beta^*, \beta \mid \alpha^*, \alpha)\alpha\bar{s}_0^{(a)}(\alpha^*, \alpha)\alpha^*\beta^*\beta. \quad (100)$$

Again the β integrals may be performed explicitly using (75) with the result

$$\begin{aligned}\langle b_0^\dagger b_t^\dagger b_t b_0\rangle &= [\bar{n}(1 - e^{-\gamma t}) + |w(t)|^2]\langle b^\dagger b\rangle_0 \\ &+ w(t)e^{i\omega_c t}e^{-(\gamma/2)t}\langle b^\dagger b b^\dagger\rangle_0 + w^*(t)e^{i\omega_c t}e^{-(\gamma/2)t}\langle b^\dagger b b\rangle_0 \\ &+ e^{-\omega t}\langle b^\dagger b b^\dagger b\rangle_0. \end{aligned} \quad (101)$$

X. Discussion

It has been the aim of this paper to present in nearly self-contained form a systematic method for computing the most important statistical properties of a system of bosons interacting weakly with a reservoir in thermal equilibrium. We have also shown in detail the results to which it leads for the simple but instructive case of the damped driven harmonic oscillator, but the method is by no means restricted to this case. The important features of this method are: 1) the reduction of the density operator master equation to a partial differential equation which may in more complicated cases be solved with numerical methods; 2) use of the conditional probability which permits arbitrary specification of the initial reduced density matrix as well as computation of two-time averages; and 3) economy of basic assumptions. Only two approximations are absolutely necessary: that the memory of the system is destroyed by its interaction with the reservoir; and that this interaction is sufficiently weak that its effects need only be considered to second order in perturbation theory.

Appendix A

In this appendix we shall derive the effects of various functions of a and $a^\dagger$ operating on the projection operator

$$\Lambda \equiv |\alpha\rangle\langle\alpha| = e^{-\alpha\alpha^*}e^{\alpha a^\dagger}|0\rangle\langle 0|e^{\alpha^* a} \equiv \Lambda^\dagger. \quad (102)$$

By (1) we see immediately that

$$a\Lambda = \alpha\Lambda, \qquad \Lambda a^\dagger = \alpha^*\Lambda. \quad (103)$$

Next, we have

$$a^\dagger\Lambda = \left(\frac{\partial}{\partial\alpha} + \alpha^*\right)\Lambda. \quad (104)$$

Proof: We have by (102)

$$a^\dagger\Lambda = e^{-\alpha\alpha^*}\frac{\partial}{\partial\alpha}(e^{\alpha a^\dagger}|0\rangle\langle 0|e^{\alpha^* a}) = e^{-\alpha\alpha^*}\frac{\partial}{\partial\alpha}(e^{\alpha\alpha^*}\Lambda) = \left(\frac{\partial}{\partial\alpha} + \alpha^*\right)\Lambda \quad \text{Q.E.D.}$$

Also,

$$\Lambda a = \left(\frac{\partial}{\partial\alpha^*} + \alpha\right)\Lambda. \quad (105)$$

This is just the adjoint of (104). Next,

$$a^\dagger a\Lambda = \left(\alpha\frac{\partial}{\partial\alpha} + \alpha\alpha^*\right)\Lambda \equiv \left(\frac{\partial}{\partial\alpha}\alpha + \alpha\alpha^* - 1\right)\Lambda. \quad (106)$$

This follows from (103) and (104). The adjoint of (106) gives

$$\Lambda a^\dagger a = \left(\alpha^*\frac{\partial}{\partial\alpha^*} + \alpha\alpha^*\right)\Lambda. \quad (107)$$

Similarly,

$$\Lambda a a^{\dagger} = \left(\alpha^* \frac{\partial}{\partial \alpha^*} + 1 + \alpha\alpha^*\right)\Lambda. \tag{108}$$

This follows from $aa^{\dagger} = a^{\dagger}a + 1$ and (107). Also,

$$a\Lambda a^{\dagger} = \alpha\alpha^*\Lambda \tag{109}$$

by (103) while

$$a^{\dagger}\Lambda a = \left[1 + \alpha\alpha^* + \alpha\frac{\partial}{\partial\alpha} + \alpha^*\frac{\partial}{\partial\alpha^*} + \frac{\partial^2}{\partial\alpha\,\partial\alpha^*}\right]\Lambda. \tag{110}$$

Proof: By (105) and (104)

$$\alpha^*\Lambda a = a^{\dagger}\left(\frac{\partial}{\partial\alpha^*} + \alpha\right)\Lambda = \left(\frac{\partial}{\partial\alpha^*} + \alpha\right)a^{\dagger}\Lambda$$
$$= \left(\frac{\partial}{\partial\alpha^*} + \alpha\right)\left(\frac{\partial}{\partial\alpha} + \alpha^*\right)\Lambda \qquad \text{Q.E.D.}$$

Appendix B

In (39), we have to evaluate

$$\begin{aligned} I &= -\frac{1}{\hbar^2}\int_{-\infty}^{t} \mathrm{Tr}_R\; V(t)V(t')s(t)f_0(R) \\ &= -\int_{-\infty}^{t} \mathrm{Tr}_R\,[G(t)a^{\dagger} + G^{\dagger}(t)a] \\ &\qquad \cdot [G(t')a^{\dagger} + G^{\dagger}(t')a]s(t)f_0\,dt' \end{aligned} \tag{111}$$

where we used (41) and

$$G(t) = \Sigma_j \kappa_j b_j e^{i(\omega_c - \omega_j)t} \tag{112}$$

$$\begin{aligned} f_0(R) &= \frac{e^{-\beta R}}{\mathrm{Tr}_R\, e^{-\beta R}} \\ &= \prod_j (1 - e^{-\beta\hbar\omega_j})e^{-\beta\hbar\omega_j b_j^+ b_j}. \end{aligned} \tag{113}$$

To begin, we note that

$$\mathrm{Tr}_R\; b_j b_k f_0(R) = 0 = \mathrm{Tr}_R\; b_j^{\dagger} b_k^{\dagger} f_0(R) \tag{114}$$

since

$$\begin{aligned} \sum_{n=0}^{\infty} \langle n|\, b e^{-\lambda b^{\dagger} b}\, |n\rangle &= 0 \\ \sum_{n=0}^{\infty} \langle n|\, b^2 e^{-\lambda b^{\dagger} b}\, |n\rangle &= 0. \end{aligned} \tag{115}$$

Next, we see that

$$\mathrm{Tr}_R\; b_j^{\dagger} b_k f_0(R) = \delta_{jk}\langle b_j^{\dagger} b_j\rangle_R = \frac{\delta_{jk}}{e^{\hbar\omega_j/kT} - 1} = \delta_{jk}\bar{n}_j \tag{116}$$

since

$$\frac{\sum_0^{\infty} \langle n|\, b^{\dagger} b e^{-\lambda b^{\dagger} b}\, |n\rangle}{\sum_0^{\infty} \langle n|\, e^{-\lambda b^{\dagger} b}\, |n\rangle} = (1 - e^{-\lambda}) \sum_0^{\infty} n e^{-\lambda n} = \frac{1}{e^{\lambda} - 1}. \tag{117}$$

Also, we have

$$\langle b_j b_k^{\dagger}\rangle_R \equiv \mathrm{Tr}_R\; b_j b_k^{\dagger} f_0(R) = (1 + \bar{n}_j)\,\delta_{jk}. \tag{118}$$

It then follows easily that

$$\begin{aligned} \langle G(t)G(t')\rangle_R &= 0 = \langle G^{\dagger}(t)G^{\dagger}(t')\rangle_R \\ \langle G(t)G^{\dagger}(t')\rangle_R &= \Sigma_j\, |\kappa_j|^2\,(1 + \bar{n}_j)e^{i(\omega_c - \omega_j)(t - t')} \\ \langle G^{\dagger}(t)G(t')\rangle_R &= \Sigma_j\, |\kappa_j|^2\,\bar{n}_j e^{-i(\omega_c - \omega_j)(t - t')}. \end{aligned} \tag{119}$$

Therefore, I in (111) becomes

$$\begin{aligned} I = -\Sigma_j\, |\kappa_j|^2 \int_{-\infty}^{t} \{&(1 + \bar{n}_j)e^{-i(\omega_j - \omega_c)(t - t')}a^{\dagger}as(t) \\ &+ \bar{n}_j e^{i(\omega_j - \omega_c)(t - t')}aa^{\dagger}s(t)\}\,dt'. \end{aligned} \tag{120}$$

We assume that the loss oscillators are closely spaced in frequency with a density $g(\omega_j)$. The above sum can then be replaced by an integral

$$\Sigma_j \to \int d\omega_j g(\omega_j).$$

If in addition, we let $t - t' = v$, (120) becomes

$$\begin{aligned} I = -\int d\omega_j g(\omega_j)\, |\kappa(\omega_j)|^2 \Big\{&(1 + \bar{n}_j)\int_0^{\infty} e^{-i(\omega_j - \omega_c)v}\,dv a^{\dagger}as \\ &+ \bar{n}_j \int_0^{\infty} e^{i(\omega_j - \omega_c)v}\,dv aa^{\dagger}s\Big\}. \end{aligned} \tag{121}$$

But[24]

$$\int_0^{\infty} e^{\pm ixv}\,dv = \pi\,\delta(x) \pm iP\frac{1}{x} \tag{122}$$

where $P1/x$ is the Cauchy principle part of $1/x$. If we neglect the principle part which gives a small imaginary term which causes a small frequency shift, (121) reduces to

$$I = -\frac{\gamma}{2}(1 + \bar{n})a^{+}as - \frac{\gamma}{2}\bar{n}aa^{+}s \tag{123}$$

where we have let

$$\begin{aligned} \gamma &\triangleq 2\pi g(\omega_j)\, |\kappa(\omega_j)|^2\, |_{\omega_j = \omega_c} \\ \bar{n} &= \bar{n}(\omega_j)\, |_{\omega_j = \omega_c} = \frac{1}{e^{\hbar\omega_c/kT} - 1}. \end{aligned} \tag{124}$$

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Nonlinear Absorption of Light: Optical Saturation of Electronic Transitions in Organic Molecules with High Intensity Laser Radiation

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***Abstract*—A review of spectroscopic properties of complex molecules is presented and used to show that a simple two-level scheme is inadequate to describe the optical bleaching of dye molecules. Experimental data are reported for the transmission of intense ruby laser radiation by several types of dyes. Rate equation analyses are carried out using steady-state solutions and iterative computer solutions; calculated bleaching curves are compared with our data for cryptocyanine. On these bases, we show that, in general, the optical bleaching process involves the removal of ground-state molecules to other states having smaller absorption cross sections at the exciting frequency, and that recovery of absorption at this frequency is characterized by a complex relaxation mechanism.**

I. Introduction

A NONLINEAR ABSORBER of light has the unique property that its optical absorption can be altered by changes in the intensity of radiation incident upon it. Several different types of organic molecules undergo optical bleaching[1]–[10] when subjected to high intensity radiation and are useful as reversible (passive) Q-switches[11] for lasers.[12]–[26] Simultaneous Q-switching and mode-locking can be achieved also when certain dye solutions are incorporated in a laser cavity.[27],[28] Saturation of absorption is an essential feature required of the dye in both applications, i.e., the transmission of the dye must be power dependent in such a way that it provides less attenuation for a high-level optical signal than for a low-level signal.

Several authors have discussed the propagation of monochromatic radiation in saturable media,[29]–[32] and the evolution of a giant pulse[33]–[37] produced with passive Q-switches, but relatively little attention has been devoted to the molecular processes involved in optical bleaching. Mechanistic considerations thus far have centered mainly around the possibility of narrow spectral *hole-burning* of molecular absorption bands.[8]–[10],[20],[23],[38],[39] The energy level structure of saturable absorbers used as

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